PROOF OF FORMULA 4.244.2

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{1-x^3}} = -\frac{\pi}{3\sqrt{3}}\left[\ln 3 + \frac{\pi}{3\sqrt{3}}\right]$$

In the proof of formula 4.253.1 it was shown that

$$\int_0^1 t^{a-1}(1-t)^{b-1}\ln t\,dt = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\left[\psi(a)-\psi(a+b)\right].$$

The change of variable $t = x^3$ gives

$$\begin{aligned}\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{1-x^3}} &= \frac{1}{9}\int_0^1 t^{-2/3}(1-t)^{-1/3}\,\ln t\,dt \\ &= \Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right)\left[\psi\left(\tfrac{1}{3}\right)-\psi(1)\right].\end{aligned}$$

The identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

and the value

$$\psi\left(\frac{1}{3}\right) = -\gamma - \frac{\pi}{2\sqrt{3}} - \frac{3}{2}\ln 3 \text{ and } \psi(1) = -\gamma,$$

now give the result.