

PROOF OF FORMULA 4.245.2

$$\int_0^1 \frac{x^{4n+3} \ln x \, dx}{\sqrt{1-x^4}} = \frac{(2n)!!}{4(2n+1)!!} \left(\ln 2 - \sum_{k=1}^{2n+1} \frac{(-1)^{k-1}}{k} \right)$$

The change of variable $t = x^4$ gives

$$\int_0^1 \frac{x^{4n+3} \ln x \, dx}{\sqrt{1-x^4}} = \frac{1}{16} \int_0^1 t^n (1-t)^{-1/2} \ln t \, dt.$$

In the proof of formula 4.253.1 it was shown that

$$\int_0^1 t^{a-1} (1-t)^{b-1} \ln t \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} [\psi(a) - \psi(a+b)].$$

Therefore

$$\int_0^1 \frac{x^{4n+3} \ln x \, dx}{\sqrt{1-x^4}} = \frac{\Gamma(n+1)\Gamma(1/2)}{16\Gamma(n+3/2)} [\psi(n+1) - \psi(n+3/2)].$$

The values

$$\begin{aligned} \Gamma(n + \tfrac{3}{2}) &= \frac{\sqrt{\pi}}{2^{n+1}} (2n+1)!! \\ \psi(n + \tfrac{3}{2}) &= -\gamma + 2 \left(\sum_{k=1}^{n+1} \frac{1}{2k-1} - \ln 2 \right) \\ \psi(n+1) &= -\gamma + \sum_{k=1}^n \frac{1}{k}, \end{aligned}$$

give the result.