PROOF OF FORMULA 4.253.3

$$\int_a^\infty x^{-\nu}(x-a)^{\mu-1}\,\ln x\,dx = a^{\mu-\nu}B(\nu-\mu,\mu)\,\left[\ln a + \psi(\nu) - \psi(\nu-\mu)\right]$$

Let $x = at$ to obtain

$$\int_a^\infty x^{-\nu}(x-a)^{\mu-1}\,\ln x\,dx = a^{\mu-\nu}\int_1^\infty t^{-\nu}(t-1)^{\mu-1}\,dt + a^{\mu-\nu}\int_1^\infty t^{-\nu}(t-1)^{\mu-1}\,\ln t\,dt.$$

The change of variables $t = 1/s$ shows that the first integral is $B(\nu-\mu,\mu)$ and the second one can be evaluated using

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,\ln t\,dt = B(x,y)\,[\psi(x) - \psi(x+y)]$$

that comes from differentiating the basic integral representation of the beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

with respect to the parameter x.