

PROOF OF FORMULA 4.255.3

$$\int_0^\infty \frac{1-x^p}{1-x^2} \ln x \, dx = \left[\frac{\pi}{2} \tan\left(\frac{\pi p}{2}\right) \right]^2$$

The change of variable $t = x^2$ gives

$$\int_0^\infty \frac{1-x^p}{1-x^2} \ln x \, dx = \frac{1}{4} \int_0^\infty \frac{t^{-1/2} \ln t}{1-t} \, dt - \frac{1}{4} \int_0^\infty \frac{t^{p/2-1/2} \ln t}{1-t} \, dt.$$

Split the integrals at $t = 1$ and make the change of variable $t \mapsto 1/t$ in the one from 1 to ∞ . This gives

$$\int_0^\infty \frac{t^{-1/2} \ln t}{1-t} \, dt = 2 \int_0^1 \frac{t^{-1/2} \ln t}{1-t} \, dt.$$

The second integral gives

$$\int_0^\infty \frac{t^{p/2-1/2} \ln t}{1-t} \, dt = \int_0^1 \frac{t^{p/2-1/2} \ln t}{1-t} \, dt + \int_0^1 \frac{t^{-p/2-1/2} \ln t}{1-t} \, dt.$$

These three integrals are evaluated using

$$\int_0^1 \frac{t^{a-1} \ln t}{1-t} \, dt = -\psi'(a)$$

to produce

$$-2\psi'\left(\frac{1}{2}\right) - \psi'\left(\frac{p+1}{2}\right) - \psi'\left(\frac{-p+1}{2}\right).$$

The result is simplified using

$$\psi'(a) + \psi'(1-a) = \frac{\pi^2}{\sin^2 \pi a}.$$