PROOF OF FORMULA 4.269.2

$$\int_0^1 \frac{dx}{\sqrt{\ln(1/x)}\,(1+x^2)} = \sqrt{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}$$

The change of variables $x = e^{-u}$ gives

$$\int_0^1 \frac{dx}{\sqrt{\ln(1/x)}\,(1+x^2)} = \int_0^\infty \frac{e^{-u}\,du}{\sqrt{u}\,(1+e^{-2u})}.$$

Expand the integrand as a geometric series to obtain

$$\int_0^\infty \frac{e^{-u}\,du}{\sqrt{u}\,(1+e^{-2u})} = \sum_{k=0}^{\infty}(-1)^k \int_0^\infty e^{-(2k+1)u}\,\frac{du}{\sqrt{u}}.$$

The changes of variables $t = (2k+1)u$ and then $t = s^2$ give the result.