PROOF OF FORMULA 4.272.10

$$\int_0^1 \left(\ln \frac{1}{x} \right)^{\mu-1} (x-1)^n \left(a + \frac{nx}{x-1} \right) x^{a-1} \, dx = \Gamma(\mu) \sum_{k=0}^{n} \frac{(-1)^k}{(a+n-k)^{\mu-1}} \binom{n}{k}$$

Observe that $a + \frac{nx}{x-1} = a + n + \frac{n}{x-1}$, therefore the integral is given by

$$(a+n) \int_0^1 \left(\ln \frac{1}{x} \right)^{\mu-1} (x-1)^n x^{a-1} dx + n \int_0^1 \left(\ln \frac{1}{x} \right)^{\mu-1} (x-1)^{n-1} x^{a-1} dx$$

$$= (a+n) \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \int_0^1 \left(\ln \frac{1}{x} \right)^{\mu-1} x^{r+a-1} dx + n \sum_{r=0}^{n-1} \binom{n-1}{r} (-1)^{n-1-r} \int_0^1 \left(\ln \frac{1}{x} \right)^{\mu-1} x^{r+a-1} dx.$$

The change of variables $x = e^{-u}$ yields

$$\int_0^1 \left(\ln \frac{1}{x} \right)^{c-1} x^{b-1} dx = \int_0^\infty u^{c-1} e^{-bu} \, du = \frac{\Gamma(c)}{b^c}$$

therefore the integral is

$$\frac{(a+n)\Gamma(\mu)}{(a+n)^\mu} + \sum_{r=0}^{n-1} \left[(a+n) \binom{n}{r} - n \binom{n-1}{r} \right] (-1)^{n-r} \frac{\Gamma(\mu)}{(r+a)^\mu}$$

and this reduces to the stated answer.