PROOF OF FORMULA 4.272.11

$$\int_0^1 \left(\ln \frac{1}{x} \right)^{n-1} \frac{1-x^m}{1-x}\, dx = (n-1)! \sum_{k=1}^{m} \frac{1}{k^n}$$

Let $t = \ln 1/x$ to obtain

$$\int_0^1 \left(\ln \frac{1}{x} \right)^{n-1} \frac{1-x^m}{1-x}\, dx = - \int_0^\infty t^{n-1} \frac{1-e^{-mt}}{1-e^t}\, dt.$$

Write the integral as

$$- \int_0^\infty t^{n-1} \frac{1-e^{-mt}}{1-e^t}\, dt = \int_0^\infty t^{n-1} e^{-mt} \frac{e^{mt}-1}{e^t-1}\, dt.$$

The rational function part of the integrand can be expanded to obtain

$$- \int_0^\infty t^{n-1} \frac{1-e^{-mt}}{1-e^{-t}}\, dt = \sum_{k=0}^{m-1} \int_0^\infty t^{n-1} e^{-(m-k)t}\, dt.$$

The change of variables $s = (m-k)t$ gives

$$\int_0^\infty t^{n-1} \frac{1-e^{-mt}}{1-e^{-t}}\, dt = - \sum_{k=0}^{m-1} \frac{1}{(m-k)^n} \int_0^\infty s^{n-1} e^{-s}\, ds.$$

The integral is recognized as $\Gamma(n) = (n-1)!$, to conclude.