PROOF OF FORMULA 4.272.14

$$\int_0^1 \left(\ln \frac{1}{x} \right)^{r-1} \frac{x^{p-1}\, dx}{(1+x^q)^s} = \Gamma(r) \sum_{k=0}^{\infty} \binom{-s}{k} \frac{1}{(p+kq)^s}$$

Start with the expansion

$$(1+x)^{-s} = \sum_{k=0}^{\infty} \binom{-k}{k} x^k$$

and write the integral as

$$\int_0^1 \left(\ln \frac{1}{x} \right)^{r-1} \frac{x^{p-1}\, dx}{(1+x^q)^s} = \sum_{k=0}^{\infty} \binom{-s}{k} \int_0^1 \left(\ln \frac{1}{x} \right)^{r-1} x^{p-1+kq}\, dx.$$

The change of variables $t = -\ln x$ gives

$$\int_0^1 \left(\ln \frac{1}{x} \right)^{r-1} \frac{x^{p-1}\, dx}{(1+x^q)^s} = \sum_{k=0}^{\infty} \binom{-s}{k} \int_0^{\infty} t^{r-1} e^{-t(kq+r)}\, dt.$$

The result follows from the change of variables $w = t(kq+r)$.