PROOF OF FORMULA 4.275.2

$$\int_0^1 \left[x - \left(\frac{1}{1-\ln x} \right)^q \right] \frac{dx}{x \, \ln x} = -\psi(q)$$

Let $t = 1 - \ln x$ to obtain

$$\int_0^1 \left[x - \left(\frac{1}{1-\ln x} \right)^q \right] \frac{dx}{x \, \ln x} = \int_1^\infty \left(e^{1-t} - t^{-q} \right) \frac{dt}{1-t}.$$

Now let $w = t - 1$ and the last integral gives

$$\int_0^1 \left[x - \left(\frac{1}{1-\ln x} \right)^q \right] \frac{dx}{x \, \ln x} = - \int_0^\infty \left(e^{-w} - (1+w)^{-q} \right) \frac{dw}{w}.$$

This integral is $-\psi(q)$ according to one of the basic integral representations of this function. It appears as entry 8.361.2.