PROOF OF FORMULA 4.291.10

$$\int_0^1 \frac{\ln(1-x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2 - G$$

Let $x = \tan\varphi$ to obtain

$$\int_0^1 \frac{\ln(1-x)}{1+x^2}\,dx = \int_0^{\pi/4} \left(\ln(\cos\varphi - \sin\varphi) - \ln\cos\varphi\right)\,d\varphi.$$

Now use $\cos\varphi - \sin\varphi = \sqrt{2}\sin(\frac{\pi}{4}-\varphi)$ to obtain

$$\int_0^1 \frac{\ln(1-x)}{1+x^2}\,dx = \frac{\pi\ln 2}{8} + \int_0^{\pi/4} \ln\sin\varphi\,d\varphi - \int_0^{\pi/4}\ln\cos\varphi\,d\varphi.$$

The values

$$\begin{aligned}\int_0^{\pi/4} \ln\sin\varphi\,d\varphi &= -\frac{\pi}{4}\ln 2 - \frac{G}{2}\\ \int_0^{\pi/4} \ln\cos\varphi\,d\varphi &= -\frac{\pi}{4}\ln 2 + \frac{G}{2}\end{aligned}$$

given in entries 4.224.2 and 4.224.5 yield the result.