PROOF OF FORMULA 4.291.7

$$\int_0^\infty \frac{\ln(1+ax)}{1+x^2}\,dx = \frac{\pi}{4}\ln(1+a^2) - \int_0^a \frac{\ln x\,dx}{1+x^2}$$

Both sides vanish as $a \to 0$, so it sufficient to check that the derivatives match. This amounts to

$$\int_0^\infty \frac{x\,dx}{(1+ax)(1+x^2)} = \frac{\pi a}{2(1+a^2)} - \frac{\ln a}{1+a^2}.$$

Expanding the left hand side in partial fractions

$$\begin{aligned}\int_0^N \frac{x\,dx}{(1+ax)(1+x^2)} &= -\frac{a}{1+a^2}\int_0^N \frac{dx}{1+ax} + \frac{a}{1+a^2}\int_0^N \frac{dx}{1+x^2} + \frac{1}{1+a^2}\int_0^N \frac{x\,dx}{1+x^2} \\ &= \frac{1}{1+a^2}\left(\ln\sqrt{1+N^2} - \ln(1+aN)\right) + \frac{a}{1+a^2}\tan^{-1} N.\end{aligned}$$

The result follows from taking $N \to \infty$.