PROOF OF FORMULA 4.296.1

$$\int_0^1 \ln(1+2x\cos t + x^2)\,\frac{dx}{x} = \frac{\pi^2}{6} - \frac{t^2}{2}$$

Denote the integral by $f(t)$ and differentiate to produce

$$f'(t) = -2\sin t \int_0^1 \frac{dx}{1+2x\cos t + x^2}.$$

Now write $1+2x\cos t + x^2 = (x+\cos t)^2 + \sin^2 t$ and let $v = x + \cos t$ to obtain

$$\begin{aligned}\int_0^1 \frac{dx}{1+2x\cos t+x^2} &= \int_{\cos t}^{1+\cos t} \frac{dv}{v^2+\sin^2 t} \\ &= \frac{1}{\sin t}\int_{\cos t/\sin t}^{(1+\cos t)/\sin t} \frac{dy}{1+y^2}.\end{aligned}$$

Evaluating the last integral gives

$$\int_0^1 \frac{dx}{1+2x\cos t+x^2} = \frac{1}{\sin t}\left[\tan^{-1}\left(\frac{1+\cos t}{\sin t}\right) - \tan^{-1}\left(\frac{\cos t}{\sin t}\right)\right] = \frac{t}{2\sin t}.$$

Therefore $f'(t)) = -t$. The value

$$f(\pi/2) = \int_0^1 \frac{\ln(1+x^2)}{x}\,dx = \frac{1}{2}\int_0^1 \frac{\ln(1+v)}{v}\,dv = \frac{\pi^2}{24}$$

obtained from entry 4.291.1 gives the final result.