PROOF OF FORMULA 4.297.6

$$\int_u^v \ln \left(\frac{v+x}{u+x} \right) \frac{dx}{x} = \frac{1}{2} \ln^2 \frac{v}{u}$$

The change of variables $x = ut$ gives, with $c = v/u$,

$$\begin{aligned} \int_u^v \ln \left(\frac{v+x}{u+x} \right) \frac{dx}{x} &= \int_1^c \ln \left(\frac{t+c}{t+1} \right) \frac{dt}{t} \\ &= \int_1^c \frac{\ln(t+c)}{t} dt - \int_1^c \frac{\ln(t+1)}{t} dt. \end{aligned}$$

The first integral is transformed by the change of variable $t = cy$ to

$$\begin{aligned} \int_1^c \frac{\ln(t+c)}{t}\, dt &= \int_{1/c}^1 \frac{\ln c + \ln(1+y)}{y}\, dy \\ &= \ln^2 c + \int_1^c \frac{\ln(1+w) - \ln w}{w}\, dw, \end{aligned}$$

after the change $w = 1/y$ to produce the last integral. This gives

$$\begin{aligned} \int_u^v \ln \left(\frac{v+x}{u+x} \right) \frac{dx}{x} &= \ln^2 c - \int_1^c \frac{\ln w}{w}\, dw \\ &= \frac{1}{2} \ln^2 c, \end{aligned}$$

as claimed.