PROOF OF FORMULA 4.351.3

$$\int_1^\infty \frac{e^{-\mu x}\ln x}{1+x}\,dx = \frac{1}{2}e^{\mu}\ \text{Ei}^2(-\mu)$$

The exponential integral is defined by

$$\text{Ei}(a) = \int_{-\infty}^{a} \frac{e^t}{t}\,dt.$$

Therefore

$$\frac{d}{da}\text{Ei}(a) = \frac{e^a}{a}.$$

Start with

$$\int_1^\infty \frac{e^{-\mu x}\ln x}{1+x}\,dx = e^{\mu}\int_1^\infty \frac{e^{-\mu(x+1)}\ln x}{1+x}\,dx.$$

Differentiate with respect to μ to obtain

$$f'(\mu) = f(\mu) - \int_1^\infty e^{-\mu x}\ln x\,dx,$$

where

$$f(\mu) = \int_1^\infty \frac{e^{-\mu x}}{1+x}\,\ln x\,dx.$$

Integration by parts shows that

$$\int_1^\infty e^{-\mu x}\ln x\,dx = \frac{1}{\mu}\text{Ei}(-\mu).$$

Therefore,

$$\frac{d}{d\mu}\left[f(\mu)e^{-\mu}\right] = -\frac{1}{\mu}e^{-\mu}\text{Ei}(-\mu) = \text{Ei}(-\mu)\frac{d}{d\mu}\text{Ei}(-\mu).$$

The result follows by integrating with respect to the parameter μ.