PROOF OF FORMULA 4.355.2

$$\int_0^\infty x(\mu x^2 - \nu x - 1)e^{-\mu x^2 + 2\nu x} \ln x \, dx = \frac{1}{4\mu} + \frac{\nu\sqrt{\pi}}{4\mu\sqrt{\mu}} e^{\nu^2/\mu}\left[1 + \text{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right]$$

Write the integral as

$$\begin{aligned}\int_0^\infty x(\mu x^2 - \nu x - 1)e^{-\mu x^2 + 2\nu x} \ln x \, dx &= -\frac{1}{2}\int_0^\infty x^2 \ln x \frac{d}{dx} e^{-\mu x^2 + 2\nu x} \, dx \\ &\quad - \int_0^\infty x e^{-\mu x^2 + 2\nu x} \ln x \, dx.\end{aligned}$$

Integrate by parts to obtain

$$\int_0^\infty x(\mu x^2 - \nu x - 1)e^{-\mu x^2 + 2\nu x} \ln x \, dx = \frac{1}{2}\int_0^\infty x e^{-\mu x^2 + 2\nu x} \, dx.$$

Let $t = \sqrt{\mu}x$ and then $s = t - \nu/\sqrt{\mu}$ to get

$$\int_0^\infty x(\mu x^2 - \nu x - 1)e^{-\mu x^2 + 2\nu x} \ln x \, dx = \frac{1}{2\mu} e^{\nu^2/\mu} \int_{-\nu/\sqrt{\mu}}^\infty (s + \nu/\sqrt{\mu}) e^{-s^2} \, ds.$$

The first term integrates to $\frac{1}{2}e^{-\nu^2/\mu}$. This gives the value $1/4\mu$ in the answer. The second integral is now written as

$$\int_0^\infty e^{-s^2} \, ds + \int_0^{\nu/\sqrt{\mu}} e^{-s^2} \, ds.$$

This is now directly expressed in terms of the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} \, ds$$

to obtain the final form of the answer.