

## The integrals in Gradshteyn and Ryzhik. Part 15: Frullani integrals

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ABSTRACT. The table of Gradshteyn and Ryzhik contains some integrals that can be reduced to the Frullani type. We present a selection of them.

### 1. Introduction

The table of integrals [3] contains many evaluations of the form

$$(1.1) \quad \int_0^\infty \frac{f(ax) - f(bx)}{x} dx = [f(0) - f(\infty)] \ln \left( \frac{b}{a} \right).$$

Expressions of this type are called *Frullani integrals*. Conditions that guarantee the validity of this formula are given in [1] and [4]. In particular, the continuity of  $f'$  and the convergence of the integral are sufficient for (1.1) to hold.

### 2. A list of examples

Many of the entries in [3] are simply particular cases of (1.1).

EXAMPLE 2.1. The evaluation of **3.434.2** in [3]:

$$(2.1) \quad \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \ln \left( \frac{b}{a} \right)$$

corresponds to the function  $f(x) = e^{-x}$ .

EXAMPLE 2.2. The change of variables  $t = e^{-x}$  in Example 2.1 yields

$$(2.2) \quad \int_0^1 \frac{t^{b-1} - t^{a-1}}{\ln t} dt = \ln \left( \frac{a}{b} \right).$$

This is **4.267.8** in [3].

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EXAMPLE 2.3. A generalization of the previous example appears as entry **3.476.1** in [3]:

$$(2.3) \quad \int_0^\infty (e^{-vx^p} - e^{-ux^p}) \frac{dx}{x} = \frac{1}{p} \ln\left(\frac{u}{v}\right).$$

This comes from Frullani's result with a simple additional scaling.

EXAMPLE 2.4. The choice

$$(2.4) \quad f(x) = \frac{e^{-qx} - e^{-px}}{x},$$

with  $p, q > 0$  satisfies  $f(\infty) = 0$  and

$$(2.5) \quad f(0) = \lim_{x \rightarrow 0} \frac{e^{-qx} - e^{-px}}{x} = p - q.$$

Then Frullani's theorem yields

$$\int_0^\infty \left( \frac{e^{-aqx} - e^{-apx}}{ax} - \frac{e^{-bqx} - e^{-bpx}}{bx} \right) \frac{dx}{x} = (p - q) \ln\left(\frac{b}{a}\right),$$

that can be written as

$$\int_0^\infty \left( \frac{e^{-aqx} - e^{-apx}}{a} - \frac{e^{-bqx} - e^{-bpx}}{b} \right) \frac{dx}{x^2} = (p - q) \ln\left(\frac{b}{a}\right).$$

This is entry **3.436** in [3].

EXAMPLE 2.5. Now choose

$$(2.6) \quad f(x) = \frac{x}{1 - e^{-x}} \exp(-ce^x).$$

Then Frullani's theorem yields entry **3.329** of [3], in view of  $f(0) = e^{-c}$  and  $f(\infty) = 0$ :

$$\int_0^\infty \left( \frac{a \exp(-ce^{ax})}{1 - e^{-ax}} - \frac{b \exp(-ce^{bx})}{1 - e^{-bx}} \right) dx = e^{-c} \ln\left(\frac{b}{a}\right).$$

EXAMPLE 2.6. The next example uses

$$(2.7) \quad f(x) = (x + c)^{-\mu},$$

with  $c, \mu > 0$ , to produce

$$(2.8) \quad \int_0^\infty \frac{(ax + c)^{-\mu} - (bx + c)^{-\mu}}{x} dx = c^{-\mu} \ln\left(\frac{b}{a}\right).$$

This is **3.232** in [3].

EXAMPLE 2.7. Entry **4.536.2** in [3] is

$$(2.9) \quad \int_0^\infty \frac{\tan^{-1}(px) - \tan^{-1}(qx)}{x} dx = \frac{\pi}{2} \ln\left(\frac{p}{q}\right).$$

This follows directly from (1.1) by choosing  $f(x) = \tan^{-1} x$ .

EXAMPLE 2.8. The function  $f(x) = \ln(a + be^{-x})$  gives the evaluation of entry **4.319.3** of [3]:

$$(2.10) \quad \int_0^\infty \frac{\ln(a + be^{-px}) - \ln(a + be^{-qx})}{x} dx = \ln\left(\frac{a}{a+b}\right) \ln\left(\frac{p}{q}\right).$$

EXAMPLE 2.9. The function  $f(x) = ab \ln(1+x)/x$  produces entry **4.297.7** of [3]:

$$(2.11) \quad \int_0^\infty \frac{b \ln(1+ax) - a \ln(1+bx)}{x^2} dx = ab \ln\left(\frac{b}{a}\right).$$

EXAMPLE 2.10. Entry **3.484** of [3]:

$$(2.12) \quad \int_0^\infty \left[ \left(1 + \frac{a}{qx}\right)^{qx} - \left(1 + \frac{a}{px}\right)^{px} \right] \frac{dx}{x} = (e^a - 1) \ln\left(\frac{q}{p}\right),$$

is obtained by choosing  $f(x) = (1 + a/x)^x$  in (1.1).

EXAMPLE 2.11. The final example in this section corresponds to the function

$$(2.13) \quad f(x) = \frac{a + be^{-x}}{ce^{px} + g + he^{-x}}$$

that produces entry **3.412.1** of [3]:

$$(2.14) \quad \int_0^\infty \left[ \frac{a + be^{-px}}{ce^{px} + g + he^{-px}} - \frac{a + be^{-qx}}{ce^{qx} + g + he^{-qx}} \right] \frac{dx}{x} = \frac{a+b}{c+g+h} \ln\left(\frac{q}{p}\right).$$

### 3. A separate source of examples

The list presented in this section contains integrals of Frullani type that were found in volume 1 of Ramanujan's Notebooks [2].

EXAMPLE 3.1.

$$\int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x} dx = \frac{\pi}{2} \ln \frac{a}{b}$$

EXAMPLE 3.2.

$$\int_0^\infty \ln \frac{p + qe^{-ax}}{p + qe^{-bx}} \frac{dx}{x} = \ln\left(1 + \frac{q}{p}\right) \ln \frac{b}{a}$$

EXAMPLE 3.3.

$$\int_0^\infty \left[ \left(\frac{ax+p}{ax+q}\right)^n - \left(\frac{bx+p}{bx+q}\right)^n \right] \frac{dx}{x} = \left(1 - \frac{p^n}{q^n}\right) \ln \frac{a}{b}$$

where  $a, b, p, q$  are all positive.

EXAMPLE 3.4.

$$\int_0^\infty \frac{\cos ax - \cos bx}{x} dx = \ln \frac{b}{a}$$

EXAMPLE 3.5.

$$\int_0^\infty \sin\left(\frac{(b-a)x}{2}\right) \sin\left(\frac{(b+a)x}{2}\right) \frac{dx}{x} = \int_0^\infty \frac{\cos ax - \cos bx}{2x} dx = \frac{1}{2} \ln \frac{b}{a}$$

EXAMPLE 3.6.

$$\int_0^\infty \sin px \sin qx \frac{dx}{x} = \int_0^\infty \frac{\cos[(p-q)x] - \cos[(p+q)x]}{2x} dx = \frac{1}{2} \ln \frac{p+q}{p-q}$$

EXAMPLE 3.7. The evaluation of

$$\int_0^\infty \ln \left( \frac{1 + 2n \cos ax + n^2}{1 + 2n \cos bx + n^2} \right) \frac{dx}{x} = \begin{cases} \ln \frac{b}{a} \ln(1+n)^2 & n^2 < 1 \\ \ln \frac{b}{a} \ln \left(1 + \frac{1}{n}\right)^2 & n^2 > 1 \end{cases}$$

is more delicate and is given in detail in the next section.

EXAMPLE 3.8. The value

$$\int_0^\infty \frac{e^{-ax} \sin ax - e^{-bx} \sin bx}{x} dx = 0$$

follows directly from (1.1) since, in this case  $f(x) = e^{-x} \sin x$  satisfies  $f(\infty) = f(0) = 0$ .

EXAMPLE 3.9.

$$\int_0^\infty \frac{e^{-ax} \cos ax - e^{-bx} \cos bx}{x} dx = \ln \frac{b}{a}.$$

#### 4. A more delicate example

Entry 4.324.2 of [3] states that

$$(4.1) \quad \int_0^\infty [\ln(1 + 2a \cos px + a^2) - \ln(1 + 2a \cos qx + a^2)] \frac{dx}{x} = \begin{cases} 2 \ln \left(\frac{q}{p}\right) \ln(1+a) & -1 < a \leq 1 \\ 2 \ln \left(\frac{q}{p}\right) \ln(1+1/a) & a < -1 \text{ or } a \geq 1. \end{cases}$$

This requires a different approach since the obvious candidate for a direct application of Frullani's theorem, namely  $f(x) = \ln(1 + 2a \cos x + a^2)$ , does not have a limit at infinity.

In order to evaluate this entry, start with

$$(4.2) \quad \int_0^1 x^y dx = \frac{1}{y+1},$$

so

$$(4.3) \quad \int_0^1 dy \int_0^1 x^y dx = \int_0^1 dx \int_0^1 x^y dy = \int_0^1 \frac{x-1}{\ln x} dx = \int_0^1 \frac{dy}{y+1} = \ln 2.$$

This is now generalized for arbitrary symbols  $\alpha$  and  $\beta$  as

$$(4.4) \quad \int_0^\infty \frac{e^{\alpha t} - e^{\beta t}}{t} dt = \ln \left(\frac{\beta}{\alpha}\right).$$

To prove (4.4), make the substitution  $u = e^{-t}$  that turns the integral into

$$\begin{aligned} \int_0^1 \frac{u^{-1-\beta} - u^{-1-\alpha}}{\ln u} du &= \int_0^1 du \int_{-1-\alpha}^{-1-\beta} u^w dw \\ &= \int_{-1-\alpha}^{-1-\beta} dw \int_0^1 u^w du \\ &= \int_{-1-\alpha}^{-1-\beta} \frac{dw}{w+1} \\ &= \ln \left( \frac{\beta}{\alpha} \right). \end{aligned}$$

Now observe that  $|\frac{2a \cos(rx)}{1+a^2}| \leq 1$ ; therefore it is legitimate to expand the logarithmic terms as infinite series using  $\ln(1+z) = \sum_k (-1)^{k-1} \frac{z^k}{k}$ . The outcome reads

$$\begin{aligned} \int_0^\infty \frac{dx}{x} \sum_{k \geq 1} \frac{(-1)^{k-1} A^k (\cos^k px - \cos^k qx)}{k} &= \\ \sum_{k \geq 1} \frac{(-1)^{k-1} A^k}{2^k k} \int_0^\infty \frac{(e^{ipx} + e^{-ipx})^k - (e^{iqx} + e^{-iqx})^k}{x} dx; \end{aligned}$$

where  $A = 2a/(1+a^2)$ . The inner integral is evaluated using some binomial expansions. That is,

$$(4.5) \quad \int_0^\infty \frac{(e^{ipx} + e^{-ipx})^k - (e^{iqx} + e^{-iqx})^k}{x} dx = \sum_{r=0}^k \binom{k}{r} \int_0^\infty \frac{e^{(2r-k)ipx} - e^{(2r-k)iqx}}{x} dx.$$

It is time to employ equation (4.4). A closer look at (4.5) shows that care must be exercised. The integrals are sensitive to the *parity* of  $k$ . More precisely, the quantity  $2r - k$  vanishes if and only if  $k$  is even and  $r = k/2$ , in which case there is a zero contribution to summation. Otherwise, the second integral in (4.5) is always equal to  $\ln(q/p)$ . Therefore,

$$\sum_{r=0}^k \binom{k}{r} \int_0^\infty [e^{(2r-k)ipx} - e^{(2r-k)iqx}] \frac{dx}{x} = \begin{cases} 2^k \ln \left( \frac{q}{p} \right) & \text{if } k \text{ is odd,} \\ \left( 2^k - \binom{k}{k/2} \right) \ln \left( \frac{q}{p} \right) & \text{if } k \text{ is even.} \end{cases}$$

Combining the results obtained thus far yields

$$\begin{aligned}
(4.6) \quad I &= \int_0^\infty \frac{\ln(1 + 2a \cos(px) + a^2) - \ln(1 + 2a \cos(qx) + a^2)}{x} dx \\
&= \int_0^\infty \frac{dx}{x} \sum_{k \geq 1} \frac{(-1)^{k-1} A^k (\cos^k px - \cos^k qx)}{k} \\
&= \sum_{k \geq 1} \frac{(-1)^{k-1} A^k}{k 2^k} \sum_{r=0}^k \binom{k}{r} \int_0^\infty \frac{e^{(2r-k)ipx} - e^{(2r-k)iqx}}{x} dx \\
&= \ln\left(\frac{q}{p}\right) \sum_{k \text{ odd}} \frac{(-1)^{k-1} A^k}{k} + \ln\left(\frac{q}{p}\right) \sum_{k \text{ even}} \frac{(-1)^{k-1} A^k}{k} \left(1 - \frac{1}{2^k} \binom{k}{k/2}\right) \\
&= \ln\left(\frac{q}{p}\right) \sum_{k \geq 1} \frac{(-1)^{k-1} A^k}{k} + \ln\left(\frac{q}{p}\right) \sum_{k \geq 1} \frac{1}{2k} \left(\frac{A}{2}\right)^{2k} \binom{2k}{k} \\
&= \ln\left(\frac{q}{p}\right) \ln(1 + A) + \frac{1}{2} \ln\left(\frac{q}{p}\right) \sum_{k \geq 1} \binom{2k}{k} \frac{1}{k} \left(\frac{A^2}{2^2}\right)^k.
\end{aligned}$$

The last step utilizes the Taylor series

$$(4.7) \quad \sum_{k \geq 1} \binom{2k}{k} \frac{Q^k}{k} = -2 \ln\left(\frac{1}{2} [1 + \sqrt{1 - 4Q}]\right)$$

This follows from the binomial series  $\sum_{k \geq 0} \binom{2k}{k} R^k = 1/\sqrt{1 - 4R}$  after rearranging in the manner

$$\sum_{k \geq 1} \binom{2k}{k} R^{k-1} = \frac{1}{R\sqrt{1 - 4R}} - \frac{1}{R} = \frac{4}{\sqrt{1 - 4R}(1 + \sqrt{1 - 4R})},$$

and then integrating by parts (from 0 to  $Q$ )

$$\sum_{k \geq 1} \binom{2k}{k} \frac{Q^k}{k} = \int_0^Q \frac{4 \cdot dR}{\sqrt{1 - 4R}(1 + \sqrt{1 - 4R})} = \int_1^{\sqrt{1-4Q}} \frac{-2 \cdot du}{1 + u} = -2 \ln\left(\frac{1}{2} [1 + \sqrt{1 - 4Q}]\right).$$

Formula (4.7) applied to equation (4.6) leads to

$$I = \ln\left(\frac{q}{p}\right) \ln(1 + A) - \ln\left(\frac{q}{p}\right) \ln\left(\frac{1}{2} [1 + \sqrt{1 - 4Q}]\right).$$

It remains to replace  $Q = A^2/2^2 = a^2/(1 + a^2)^2$  and use the identity

$$1 - 4Q = \frac{(a^2 - 1)^2}{(a^2 + 1)^2}.$$

Observe that the expression for  $\sqrt{1 - 4Q}$  depends on whether  $|a| > 1$  or not. The proof is complete.

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