ON SOME FAMILIES OF INTEGRALS CONNECTED TO THE HURWITZ ZETA FUNCTION

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Abstract. Expressions for a family of integrals involving the Hurwitz zeta function are established using standard properties of the Fourier transform.

1. Introduction

The Hurwitz zeta function, defined by

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(a+n)^s} \tag{1.1}$$

for $s\in\mathbb{C}$, $\operatorname{Re}s>1$, and $a\neq 0,-1,-2,\cdots$, admits the integral representation

$$\zeta(s,a) \;=\; \frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-at}}{1-e^{-t}}t^{s-1}dt, \tag{1.2}$$

where $\Gamma(s)$ is Euler's gamma function, which is valid for $\operatorname{Re}s>1$ and $\operatorname{Re}a>0$, and can be used to prove that $\zeta(s,a)$ admits an analytic extension to the whole complex plane except for a simple pole at $s=1$. Hermite proved an alternate integral representation, which actually provides an explicit realization of this analytic continuation to $\mathbb{C}-\{1\}$ and $\operatorname{Re}a>0$:

$$\zeta(s,a) \;=\; \frac{a^{-s}}{2}+\frac{a^{1-s}}{s-1}+2\int_0^\infty \frac{\sin(s\tan^{-1}(t/a))\,dt}{(a^2+t^2)^{s/2}\,(e^{2\pi t}-1)}. \tag{1.3}$$

The function $\zeta(s,a)$ is analytic for $s\neq 1$, and direct differentiation of (1.3) yields

$$\begin{aligned}\zeta'(s,a)&=-\frac{1}{2}a^{-s}\ln a-\frac{a^{1-s}}{(s-1)^2}-\frac{a^{1-s}}{s-1}\ln a-2a^{1-s}\ln a\int_0^\infty\frac{\sin(s\tan^{-1}t)\,dt}{(1+t^2)^{s/2}(e^{2\pi at}-1)}\\&+2a^{1-s}\int_0^\infty\frac{\cos(s\tan^{-1}t)\,\tan^{-1}t\,dt}{(1+t^2)^{s/2}(e^{2\pi at}-1)}-a^{1-s}\int_0^\infty\frac{\sin(s\tan^{-1}t)\,\ln(1+t^2)\,dt}{(1+t^2)^{s/2}(e^{2\pi at}-1)},\end{aligned} \tag{1.4}$$

where $\zeta'(s,a)$ denotes $\partial\zeta(s,a)/\partial s$.

The work presently discussed is a continuation of [2, 4, 7] where these integral representations have been employed to evaluate interesting definite integrals. General information about $\zeta(s,a)$ can be found in [1], [5] and [6].

The main result is presented next.

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Theorem 1.1. *Let $n \in \mathbb{N}_0$. For $\operatorname{Re} a > 0$ and $0 \leq 2n < \operatorname{Re} s$, define*

$$S_n(a,s) := \int_0^\infty \frac{t^{2n} \sin(s \tan^{-1}(t/a))\, dt}{(a^2+t^2)^{s/2}(e^{2\pi t}-1)}. \tag{1.5}$$

Then

$$S_n(a,s) = \frac{1}{2} \sum_{m=0}^{2n} (-1)^{m+n} \binom{2n}{m} a^m P_1(a, m+s-2n), \tag{1.6}$$

where

$$P_1(a,s) = \zeta(s,a) - \frac{a^{-s}}{2} - \frac{a^{1-s}}{s-1}. \tag{1.7}$$

Observe that (1.3) corresponds to the special case $n=0$ in (1.5).

The proof of Theorem 1.1 is based on identifying the Fourier sine transform of two special functions and then apply the corresponding Parseval identity. Recall that for a function defined on the half-line, the *Fourier sine transform* is

$$\mathcal{F}(f)(\omega) := \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin(\omega t)\, dt, \tag{1.8}$$

provided the integral converges. The corresponding Parseval identity states that

$$\int_0^\infty \mathcal{F}(f)(\omega)\mathcal{F}(g)(\omega)\, d\omega = \int_0^\infty f(t)g(t)\, dt. \tag{1.9}$$

Theorem 1.1 is a direct consequence of Parseval's relation applied to the functions

$$f(t) = 1/(e^{2\pi t}-1) \text{ and } g(t) = \frac{t^{2n} \sin\left[s \tan^{-1}(t/a)\right]}{(a^2+t^2)^{s/2}}. \tag{1.10}$$

The Fourier sine transform of f comes from entry 3.951.12 of [3]. It states an equivalent form of the identity

$$\int_0^\infty \frac{\sin(\omega t)\, dt}{e^{2\pi t}-1} = \frac{1}{2}\left(\frac{1}{e^\omega - 1} + \frac{1}{2} - \frac{1}{\omega} \right). \tag{1.11}$$

The Fourier sine transform of $g(t)$ is given in terms of the *associated Laguerre polynomials* $L_n^k(x)$ defined by the Rodrigues representation

$$L_n^k(x) = \frac{e^x x^{-k}}{n!} \frac{d^n}{dx^n}\left(e^{-x}x^{n+k}\right), \quad n \in \mathbb{N} \cup \{0\}. \tag{1.12}$$

Theorem 1.1 is extended in Section 3 to include integrals in which the kernel $1/(e^{2\pi t}-1)$ is replaced by

$$\frac{1}{e^{\pi t}+1}, \ \frac{1}{\sinh \pi t}, \text{ or } \frac{1}{\cosh \pi t}. \tag{1.13}$$

Consider the families of integrals

$$\begin{aligned} I_k(q) &= \int_0^\infty \frac{t\, dt}{(1+t^2)^{k+1}\,(e^{2\pi q t}-1)}, \\ T_k(q) &= \int_0^\infty \frac{t^k \, \tan^{-1} t}{e^{2\pi q}-1}\, dt, \\ L_k(q) &= \int_0^\infty \frac{t^k \, \ln(1+t^2)}{e^{2\pi q}-1}\, dt. \end{aligned} \tag{1.14}$$

The reader will find in [2] explicit expression for $I_k(q)$ in terms of the derivatives of the polygamma function and for $T_{2k}(q)$ and $L_{2k+1}(q)$ in terms of derivatives of the Hurwitz zeta function. Explicit expressions for $T_{2k+1}(q)$ and $L_{2k}(q)$ remains an open problem. Future work will analyze the evaluations discussed here and thier relation to this problem.

2. The proof

The proof of Theorem 1.1 is based on the computation of two Fourier sine transforms. Formula 3.951.12 in [3] states an equivalent form of the identity

$$\int_0^\infty \frac{\sin(\omega t)\, dt}{e^{2\pi t}-1} = \frac{1}{2}\left(\frac{1}{e^{\omega}-1}+\frac{1}{2}-\frac{1}{\omega}\right) \tag{2.1}$$

which gives the sine transform of

$$f(t) = \frac{1}{e^{2\pi t}-1} \tag{2.2}$$

as

$$\mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{e^{\omega}-1}+\frac{1}{2}-\frac{1}{\omega}\right). \tag{2.3}$$

The second Fourier sine transform is that of the associated Laguerre polynomials (1.12). The explicit formula

$$L_n^k(x) = \sum_{j=0}^{n} \frac{(-1)^j\,(n+k)!}{(n-j)!\,(k+j)!\,j!}x^j \tag{2.4}$$

is employed in the derivation.

Formula 3.769.4 of [3] contains the integral representation

$$\int_0^\infty t^{2n}\left[(a-it)^{-s}-(a+it)^{-s}\right]\sin\omega t\, dt = \frac{(-1)^n i\pi\,(2n)!}{\Gamma(s)\, e^{a\omega}\omega^{2n+1-s}}L_{2n}^{s-2n-1}(a\omega), \tag{2.5}$$

for $\omega>0$, $\operatorname{Re} a>0$ and $0\leq 2n<\operatorname{Re} s$. The integrand can be simplified using

$$(a-it)^{-s}-(a+it)^{-s} = \frac{(a+it)^s-(a-it)^s}{(a^2+t^2)^s} = \frac{2i\sin(s\alpha)}{(a^2+t^2)^{s/2}}, \tag{2.6}$$

where $\alpha=\tan^{-1}(t/a)$. Therefore (2.5) can be written as

$$\int_0^\infty \frac{t^{2n}\sin\left[s\tan^{-1}(t/a)\right]}{(a^2+t^2)^{s/2}}\sin(\omega t)\, dt = \frac{(-1)^n\pi\,(2n)!}{2\Gamma(s)\, e^{a\omega}\omega^{2n+1-s}}L_{2n}^{s-2n-1}(a\omega).$$

Therefore the Fourier sine transform of

$$g(t) = \frac{t^{2n}\sin\left[s\tan^{-1}(t/a)\right]}{(a^2+t^2)^{s/2}} \tag{2.7}$$

is given by

$$\mathcal{F}(g)(\omega) = \frac{(-1)^n\pi\,(2n)!}{2\Gamma(s)\, e^{a\omega}\omega^{2n+1-s}}L_{2n}^{s-2n-1}(a\omega). \tag{2.8}$$

Parseval's identity (1.9) gives the next result.

Lemma 2.1. For $\operatorname{Re} a,\ s > 0$,

$$\int_0^\infty \frac{t^{2n} \sin\left[s \tan^{-1}(t/a) \right] \, dt}{(t^2+a^2)^{s/2} \, (e^{2\pi t}-1)} = \frac{(-1)^n (2n)!}{2\Gamma(s)} \int_0^\infty \frac{L_{2n}^{s-2n-1}(a\omega)}{\omega^{2n+1-s} \, e^{a\omega}} \left(\frac{1}{e^\omega - 1} + \frac{1}{2} - \frac{1}{\omega} \right) d\omega.$$

The explicit formula (2.4) for the Laguerre polynomials is now employed to evaluate the integral on the right hand side of the previous Lemma.

The integrand contains the term

$$L_{2n}^{s-2n-1}(a\omega) = \sum_{j=0}^{2n} \frac{(-1)^j (s-1)! a^j \omega^j}{(2n-j)!(s-2n-1+j)! j!} \tag{2.9}$$

that yields

$$\begin{aligned}
&\int_0^\infty \frac{L_{2n}^{s-2n-1}(a\omega) \, \omega^{s-2n-1} e^{-a\omega} \, d\omega}{e^\omega - 1} \\
&= \sum_{j=0}^{2n} \frac{(-1)^j (s-1)! a^j}{(2n-j)!(s-2n-1+j)! j!} \int_0^\infty \frac{\omega^{s-2n-1+j} e^{-(a+1)\omega}}{1-e^{-\omega}} d\omega \\
&= \sum_{j=0}^{2n} \frac{(-1)^j (s-1)! a^j}{(2n-j)! j!} (s-2n-j) \zeta(s-2n+j, a+1).
\end{aligned}$$

In the last step we have employed the integral representation for the Hurwitz zeta function

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\omega^{s-1} e^{-a\omega} \, d\omega}{1-e^{-\omega}}. \tag{2.10}$$

The other two types of integrals, namely those corresponding to the term $1/2$ and $1/\omega$ are elementary. This establishes the result of Theorem 1.1.

3. Related integrals

In this section we produce results similar to Theorem 1.1 for a family of integrals of the form

$$\int_0^\infty f(t) K(t) \, dt, \tag{3.1}$$

where the kernel $1/(e^{2\pi t}-1)$ in Theorem 1.1 is replaced by

$$\frac{1}{e^{\pi t}+1}, \ \frac{1}{\sinh \pi t}, \ \text{or} \ \frac{1}{\cosh \pi t}. \tag{3.2}$$

The next lemma provides relates the previous kernels to the Hurwitz zeta function. This is the main tool for the explicit evaluations of (3.1).

Lemma 3.1. Assume $\operatorname{Re} s > 1$ and $\operatorname{Re} a \geq 0$. Then

$$\int_0^\infty \frac{t^{s-1} e^{-at} \, dt}{\sinh t} = \Gamma(s) \left[\zeta(s,a) - 2^{-s} \zeta(s, a/2) \right]. \tag{3.3}$$

If $\operatorname{Re} a > 0$, then

$$\int_0^\infty \frac{t^{s-1} e^{-at} \, dt}{1+e^{-t}} = \Gamma(s) \left[2^{1-s} \zeta(s, a/2) - \zeta(s,a) \right], \tag{3.4}$$

and

$$\int_0^\infty \frac{t^{s-1}e^{-at}\,dt}{\cosh t} = \Gamma(s)2^{-2s}\left[\zeta(s,\tfrac{a+1}{4}) - \zeta(s,\tfrac{a+3}{4})\right], \tag{3.5}$$

These integrals are well-known variations of (2.10). Details are in [2].

Theorem 3.2. Let $n \in \mathbb{N}_0$. For $\operatorname{Re} a > 0$ and $0 \leq 2n < \operatorname{Re} s$, define

$$SH_n(a,s) := \int_0^\infty \frac{t^{2n}\sin(s\tan^{-1}(t/a))\,dt}{(a^2+t^2)^{s/2}\sinh \pi t}. \tag{3.6}$$

Then

$$SH_n(a,s) = \frac{1}{2}\sum_{m=0}^{2n}(-1)^{m+n}\binom{2n}{m}a^m P_2(a, m+s-2n), \tag{3.7}$$

where

$$P_2(a,s) = 2^{2-s}\zeta(s,a/2) - 2\zeta(s,a) - a^{-s}. \tag{3.8}$$

Proof. The identity

$$\int_0^\infty \frac{\sin \omega t}{\sinh \beta t}\,dt = \frac{\pi}{2\beta}\tanh\frac{\pi\omega}{2\beta} \tag{3.9}$$

appears as entry 3.981.1 in [3].

The value $\beta = \pi$ in (3.9) shows that the sine Fourier transform of $1/\sinh \pi t$ is $\frac{1}{2}\tanh(\omega/2)$. Then (1.9) and (2.5) give

$$\begin{aligned} &\int_0^\infty \frac{t^{2n}\sin(s\tan^{-1}(t/a))\,dt}{(a^2+t^2)^{s/2}\,\sinh\pi t} \\ &= \frac{(-1)^n\pi(2n)!}{4\Gamma(s)}\int_0^\infty \frac{\tan(\frac{\omega}{2})L_{2n}^{s-2n-1}(a\omega)}{\omega^{2n+1-s}\,e^{a\omega}}\,d\omega. \end{aligned}$$

Now use

$$\tanh\frac{\omega}{2} = \frac{2}{1+e^{-\omega}} - 1 \tag{3.10}$$

and proceed as in the proof of Theorem 1.1. □

The next results are established along the same lines of the proof presented above. The details are omitted. Entries 3.911.1 and 3.981.3 in [3] are

$$\int_0^\infty \frac{\sin\omega t}{e^{\beta t}+1}\,dt = \frac{1}{2\omega} - \frac{\pi}{2\beta\sinh(\pi\omega/\beta)} \tag{3.11}$$

and

$$\int_0^\infty \frac{\cos\omega t}{\cosh\beta t}\,dt = \frac{\pi}{2\beta\cosh(\pi\omega/2\beta)}, \tag{3.12}$$

respectively. These are used instead of (3.9) in the proofs.

Theorem 3.3. Let $n \in \mathbb{N}_0$. For $\operatorname{Re} a > 0$ and $0 \leq 2n < \operatorname{Re} s$, define

$$EP_n(a,s) := \int_0^\infty \frac{t^{2n}\sin(s\tan^{-1}(t/a))\,dt}{(a^2+t^2)^{s/2}(e^{\pi t}+1)}. \tag{3.13}$$

Then

$$EP_n(a,s) = \frac{1}{2}\sum_{m=0}^{2n}(-1)^{m+n}\binom{2n}{m}a^m P_3(a, m+s-2n), \tag{3.14}$$

where

$$P_3(a,s) = \frac{a^{1-s}}{s-1} - \zeta(s,a) - 2^{-s}\zeta(s,a/2). \tag{3.15}$$

Theorem 3.4. Let $n \in \mathbb{N}_0$. For $\operatorname{Re} a > 0$ and $0 \leq 2n < \operatorname{Re} s$, define

$$CH_n(a,s) := \int_0^\infty \frac{t^{2n}\cos(s\tan^{-1}(t/a))\,dt}{(a^2+t^2)^{s/2}\cosh(\pi t/2)}. \tag{3.16}$$

Then

$$CH_n(a,s) = \frac{1}{2}\sum_{m=0}^{2n}(-1)^{m+n}\binom{2n}{m}a^m P_4(a, m+s-2n), \tag{3.17}$$

where

$$P_4(a,s) = \frac{1}{2^{2s}}\left[\zeta(s,\tfrac{a+1}{4}) - \zeta(s,\tfrac{a+3}{4})\right]. \tag{3.18}$$

The final result describes integrals containing odd powers of t in the integrand. As before, the proofs are similar to that of Theorem 1.1, so they are omitted.

Theorem 3.5. Let $n \in \mathbb{N}_0$. Assume for $\operatorname{Re} a > 0$ and $-1 \leq 2n+1 < \operatorname{Re} s$. Then

$$\int_0^\infty \frac{t^{2n+1}\cos\left(s\tan^{-1}(t/a)\right)\,dt}{(a^2+t^2)^{s/2}\,(e^{2\pi t}-1)} = \frac{1}{2}\sum_{m=0}^{2n+1}(-1)^{m+n}\binom{2n+1}{m}a^m P_1(m+s-2n-1) \tag{3.19}$$

$$\int_0^\infty \frac{t^{2n+1}\cos\left(s\tan^{-1}(t/a)\right)\,dt}{(a^2+t^2)^{s/2}\,\sinh\pi t} = \frac{1}{2}\sum_{m=0}^{2n+1}(-1)^{m+n}\binom{2n+1}{m}a^m P_2(m+s-2n-1) \tag{3.20}$$

$$\int_0^\infty \frac{t^{2n+1}\cos\left(s\tan^{-1}(t/a)\right)\,dt}{(a^2+t^2)^{s/2}\,(e^{\pi t}+1)} = \frac{1}{2}\sum_{m=0}^{2n+1}(-1)^{m+n}\binom{2n+1}{m}a^m P_3(m+s-2n-1) \tag{3.21}$$

and if $\operatorname{Re} a > 0$ and $0 \leq 2n < \operatorname{Re} s - 1$,

$$\int_0^\infty \frac{t^{2n+1}\sin\left(s\tan^{-1}(t/a)\right)\,dt}{(a^2+t^2)^{s/2}\,\cosh(\pi t/2)} = \frac{1}{2}\sum_{m=0}^{2n+1}(-1)^{m+n}\binom{2n+1}{m}a^m P_4(m+s-2n-1). \tag{3.22}$$

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