The 2-adic valuation of Stirling numbers

Tewodros Amdeberhan, Dante Manna and Victor H. Moll
Department of Mathematics, Tulane University
New Orleans, LA 70118

Abstract

We analyze properties of the 2-adic valuations of the Stirling numbers of the second kind.

1 Introduction

Divisibility properties of integer sequences have always been objects of interest for number theorists. Nowadays these are expressed in terms p-adic valuations. Given a prime p and a positive integer m, there exist unique integers a, n, with a not divisible by p and $n \geq 0$, such that $m = ap^n$. The number n is called the *p-adic valuation* of m, denoted by $n = \nu_p(m)$. Thus, $\nu_p(m)$ is the highest power of p that divides m. The graph in figure 1 shows the function $\nu_2(m)$.

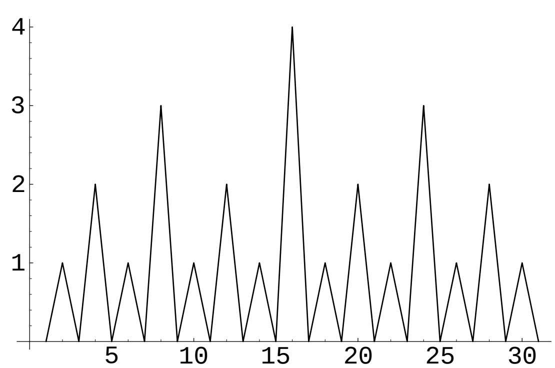

Figure 1: The 2-adic valuation of m

One of the most celebrated examples is the expression for the p-adic valuation of factorials. This is due to Legendre [8], who established

$$\nu_p(m!) = \frac{m - s_p(m)}{p-1}. \tag{1.1}$$

Here $s_p(m)$ is the sum of the base p-digits of m. In particular,

$$\nu_2(m!) = m - s_2(m). \tag{1.2}$$

The reader will find in [7] details about this identity. Figure 2 shows the graph of $\nu_2(m!)$ exhibiting its linear growth: $\nu_2(m!) \sim m$. This can be verified using the binary expansion of m:

$$m = a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + \ldots + a_r \cdot 2^r, \quad \text{with} \quad a_j \in \{0, 1\},\, a_r \neq 0,$$

so that $2^r \leq m < 2^{r+1}$. Therefore $s_2(m) = O(\log_2(m))$ and we have

$$\lim_{m \to \infty} \frac{\nu_2(m!)}{m} = 1. \tag{1.3}$$

The error term $s_2(m) = m - \nu_2(m!)$, shown in Figure 2, shows a regular pattern.

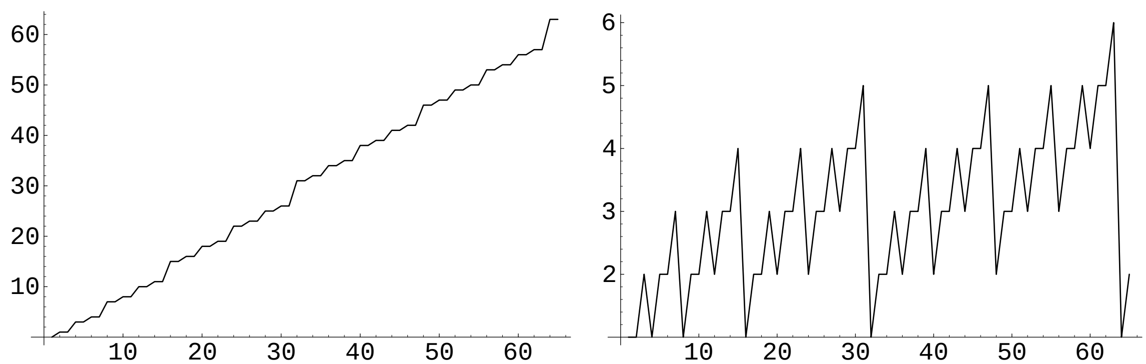

Figure 2: The 2-adic valuation of $m!$.

Legendre's result (1.2) provides an elementary proof of Kummer's identity

$$\nu_2\left(\binom{m}{k}\right) = s_2(k) + s_2(m-k) - s_2(m). \tag{1.4}$$

Not many explicit identities of this type are known.

The function ν_p is extended to $\mathbb{Q}$ by defining $\nu_p\left(\frac{a}{b}\right) = \nu_p(a) - \nu_p(b)$. The *$p$-adic metric* is then defined by

$$|r|_p := p^{-\nu_p(r)}, \quad \text{for } r \in \mathbb{Q}. \tag{1.5}$$

It satisfies the ultrametric inequality

$$|r_1 + r_1|_p \leq \text{Max}\left\{|r_1|_p\,, |r_2|_p\right\}. \tag{1.6}$$

The completion of $\mathbb{Q}$ under this metric, denoted by $\mathbb{Q}_p$, is the field of p-adic numbers. The set $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ is the ring of p-adic integers. It plays the role of $\mathbb{Z}$ inside $\mathbb{Q}_p$.

Our interest in 2-adic valuations started with the sequence

$$b_{l,m} := \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}, \tag{1.7}$$

for $m \in \mathbb{N}$ and $0 \leq l \leq m$, that appeared in the evaluation of the definite integral

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}. \tag{1.8}$$

In [2], it was shown that the polynomial defined by

$$P_m(a) := 2^{-2m} \sum_{l=0}^{m} b_{l,m} a^l \tag{1.9}$$

satisfies

$$P_m(a) = 2^{m+3/2}\,(a+1)^{m+1/2} N_{0,4}(a;m)/\pi. \tag{1.10}$$

The reader will find in [3] more details on this integral.

The results on the 2-adic valuations of $b_{l,m}$ are expressed in terms of

$$A_{l,m} := \frac{l!\,m!}{2^{m-l}} b_{l,m}. \tag{1.11}$$

The coefficients $A_{l,m}$ can be written as

$$A_{l,m} = \alpha_l(m) \prod_{k=1}^{m} (4k-1) - \beta_l(m) \prod_{k=1}^{m} (4k+1), \tag{1.12}$$

for some polynomials α_l, β_l with integer coefficients and of degree l and $l-1$, respectively. The next remarkable property was conjectured in [4] and established by J. Little in [10].

Theorem 1.1. *All the zeros of $\alpha_l(m)$ and $\beta_l(m)$ lie on the vertical line* $\operatorname{Re} m = -\frac{1}{2}$.

The next theorem, presented in [1], gives 2-adic properties of $A_{l,m}$.

Theorem 1.2. *The 2-adic valuation of $A_{l,m}$ satisfies*

$$\nu_2(A_{l,m}) = \nu_2(\,(m+1-l)_{2l}\,) + l, \tag{1.13}$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ is the Pochhammer symbol.

The identity

$$(a)_k = \frac{(a+k-1)!}{(a-1)!} \tag{1.14}$$

and Legendre's identity (1.2) yields the next expression for $\nu_2(A_{l,m})$.

Corollary 1.3. *The 2-adic valuation of $A_{l,m}$ is given by*

$$\nu_2(A_{l,m}) = 3l - s_2(m+l) + s_2(m-l). \tag{1.15}$$

Among the other examples of 2-adic valuations, we mention the results of H. Cohen [6] on the partial sums of the polylogarithmic series

$$\mathrm{Li}_k(x) := \sum_{j=1}^{\infty} \frac{x^j}{j^k}. \tag{1.16}$$

Cohen proves that the sum[1]

$$L_k(n) := \sum_{j=1}^{n} \frac{2^j}{j^k} \tag{1.17}$$

satisfies

$$\nu_2(L_1(2^m)) = 2^m + 2m - 4, \text{ for } m \geq 4, \tag{1.18}$$

and

$$\nu_2(L_2(2^m)) = 2^m + m - 1, \text{ for } m \geq 4. \tag{1.19}$$

The graph in figure 3 shows the linear growth of $\nu_2(L_1(m))$ and the error term $\nu_2(L_1(m)) - m$.

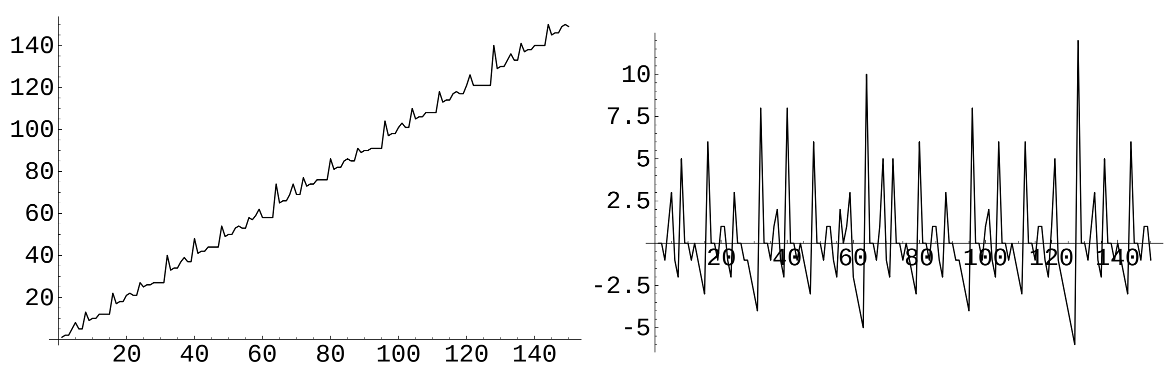

Figure 3: The 2-adic valuation of $L_1(m)$

In this paper we analyze the 2-adic valuation of the Stirling numbers of the second kind $S(n,k)$, defined for $n \in \mathbb{N}$ and $0 \leq k \leq n$ as the number of ways to partition a set of n elements into exactly k nonempty subsets. Figure 4 shows the function $\nu_2(S(n,k))$ for $k = 75$ and $k = 126$. These graphs indicate the complexity of this problem. Section 6 gives a larger selection of these type of pictures.

Main conjecture. We describe an algorithm that leads to a description of the function $\nu_2(S(n,k))$. Figure 4 shows the graphs of this function for $k = 75$ and $k = 126$. The conjecture is stated here and the special case $k = 5$ is established in section 4.

[1] Cohen uses the notation $s_k(n)$, employed here in a different context.

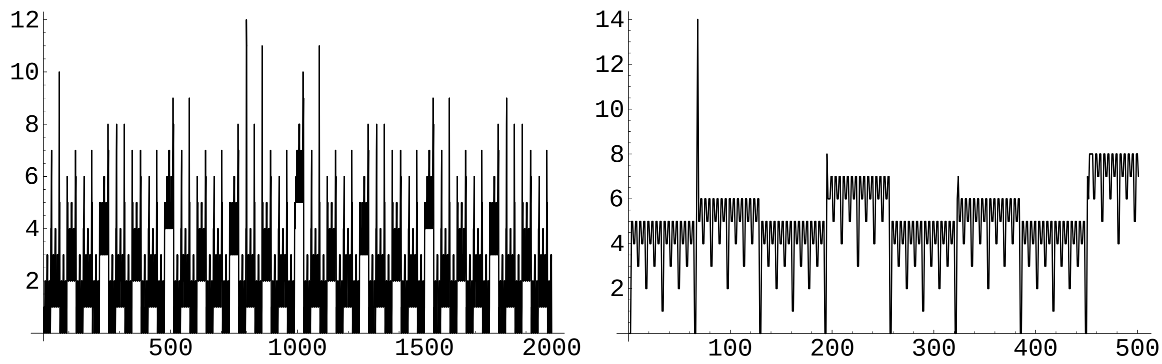

Figure 4: The data for $k = 75$ and $k = 126$.

Definition 1.4. Let $k \in \mathbb{N}$ be fixed and $m \in \mathbb{N}$. Then for $0 \leq j < 2^m$ define

$$C_{m,j} := \{2^m i + j : \quad \text{for } i \in \mathbb{N} \quad \text{such that } 2^m i + j \geq k\}. \tag{1.20}$$

For example, for $k = 5, m = 6$ and $j = 28$ we have

$$C_{6,28} = \{2^6 i + 28 : \quad i \geq 0\}. \tag{1.21}$$

We use the notation

$$\nu_2(C_{m,j}) = \{\nu_2\left(S(2^m i + j, k)\right) : i \in \mathbb{N} \text{ and } 2^m + i \geq k\}. \tag{1.22}$$

The classes $C_{m,j}$ form a partition of $\{n \in \mathbb{N} : n \geq k\}$ into classes modulo 2^m. For example, for $m = 2$ and $k = 5$, we have the four classes

$$C_{2,0} = \{2^2 i : i \in \mathbb{N}, i \geq 2\}, \qquad C_{2,1} = \{2^2 i + 1 : i \in \mathbb{N}\},$$
$$C_{2,2} = \{2^2 i + 2 : i \in \mathbb{N}\}, \qquad C_{2,3} = \{2^2 i + 3 : i \in \mathbb{N}\}.$$

The class $C_{m,j}$ is called *constant* if $\nu_2(C_{m,j})$ consists of a single value. This single value is called the constant of the class $C_{m,j}$.

For example, Corollary 3.2 shows that $\nu_2(S(4i+1, 5)) = 0$, independently of i. Therefore, the class $C_{2,1}$ is constant. Similarly $C_{2,2}$ has constant valuation 0.

We now introduce inductively the concept of *m-level*. For $m = 1$, the 1-level consists of the two classes

$$C_{1,0} = \{2i : i \in \mathbb{N}, 2i \geq k\} \text{ and } C_{1,1} = \{2i+1 : i \in \mathbb{N}, 2i+1 \geq k\}, \tag{1.23}$$

that is, the even and odd integers greater or equal than k. Assume that the $(m-1)-$level has been defined and it consists of the s classes

$$C_{m-1,i_1}, C_{m-1,i_2}, \cdots, C_{m-1,i_s}. \tag{1.24}$$

Each class C_{m-1,i_j} splits into two classes modulo 2^m, namely, C_{m,i_j} and $C_{m,i_j+2^{m-1}}$. The m-level is formed by the non-constant classes modulo 2^m.

Example. We describe the case of Stirling numbers $S(n, 10)$. Start with the fact that the 4-level consists of the classes $C_{4,7}$, $C_{4,8}$, $C_{4,9}$ and $C_{4,14}$. These split into the eight classes

$$C_{5,7},\, C_{5,23},\, C_{5,8},\, C_{5,24},\, C_{5,9},\, C_{5,25},\, C_{5,14}, \text{ and } C_{5,30},$$

modulo 32. Then one checks that $C_{5,23}$, $C_{5,24}$, $C_{5,25}$ and $C_{5,30}$ are all constant (with constant value 2 for each of them). The other four classes form the 5-level:

$$\{C_{5,7},\, C_{5,8},\, C_{5,9},\, C_{5,14}\}. \tag{1.25}$$

We are now ready to state our main conjecture.

Conjecture 1.5. *Let $k \in \mathbb{N}$ be fixed. Define $m_0 = m_0(k) \in \mathbb{N}$ by $2^{m_0-1} < k \leq 2^{m_0}$. Then the 2-adic valuation of the Stirling numbers of the second kind $S(n,k)$ satisfies:*
1) The first index for which there is a constant class is $m_0 - 1$. That is, every class is part of the j-level for $1 \leq j \leq m_0 - 2$.
2) For any $m \geq m_0$, the m-level consists of 2^{m_0-2} classes. Each one of them produces a single non-constant class for the $(m+1)$-level, keeping the number of classes at each level constant.

Example. We illustrate this conjecture for $S(n,11)$. Here $m_0 = 4$ in view of $2^3 < 11 \leq 2^4$. The conjecture predicts that the 3-level is the first with constant classes and that each level after that has exactly four classes. First of all, the four classes $C_{2,0}$, $C_{2,1}$, $C_{2,2}$, $C_{2,3}$ have non-constant 2-adic valuation. Thus, they form the 2-level. To compute the 3-level, we observe that

$$\nu_2(C_{3,3}) = \nu_2(C_{3,5}) = \{0\} \text{ and } \nu_2(C_{3,4}) = \nu_2(C_{3,6}) = \{1\},$$

so there are four constant classes. The remaining four classes $C_{3,0}$, $C_{3,1}$, $C_{3,2}$ and $C_{3,7}$ form the 3-level. Observe that each of the four classes from the 2-level splits into a constant class and a class that forms part of the 3-level.

This process continues. At the next step, the classes of the 3-level split in two giving a total of 8 classes modulo 2^4. For example, $C_{3,2}$ splits into $C_{4,2}$ and $C_{4,10}$. The conjecture states that *exactly* one of these classes has constant 2-adic valuation. Indeed, the class $C_{4,2}$ satisfies $\nu_2(C_{4,2}) \equiv 2$.

Figure 5 illustrates this process.

Elementary formulas. Throughout the paper we will use several elementary properties of $S(n,k)$, listed below:

- Relation to Pochhammer
$$x^n = \sum_{k=0}^{n} S(n,k)(x-k+1)_k \tag{1.26}$$

- An explicit formula
$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}(k-i)^n \tag{1.27}$$

- The generating function
$$\frac{1}{(1-x)(1-2x)(1-3x)\cdots(1-kx)} = \sum_{n=1}^{\infty} S(n,k)x^n \tag{1.28}$$

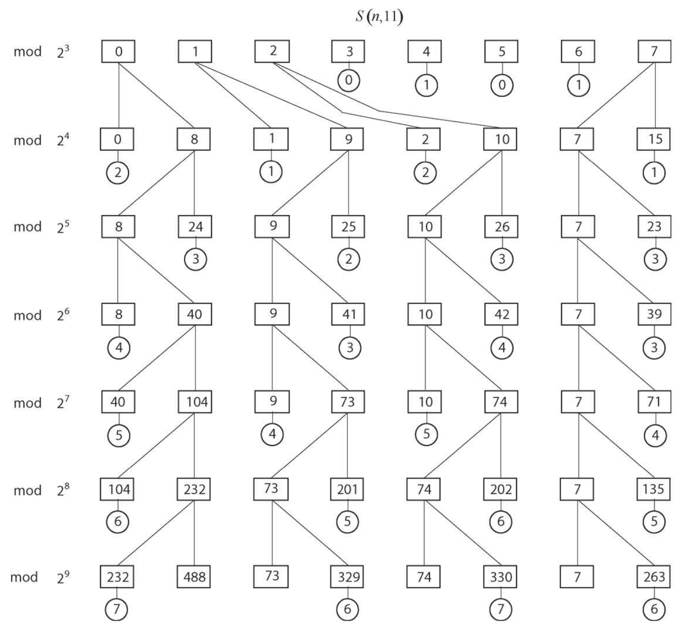

Figure 5: The splitting for $k = 11$

- The recurrence
$$S(n,k) = S(n-1,k-1) + kS(n-1,k) \tag{1.29}$$

Lengyel [9] conjectured, and De Wannemacker [12] proved, a special case of the 2-adic valuation of $S(n,k)$:
$$\nu_2\left(S(2^n,k)\right) = s_2(k) - 1, \tag{1.30}$$
independently of n. Here $s_2(k)$ is the sum of the binary digits of k. Legendre's result (1.2) then shows that
$$\nu_2(S(2^n,k)) = \nu_2(2^{k-1}/k!). \tag{1.31}$$
From here we obtain
$$\nu_2\left(S(2^n+1,k+1) = s_2(k) - 1\right. \tag{1.32}$$
as a companion of (1.30). Indeed, the recurrence (1.29) yields
$$S(2^n+1,k+1) = S(2^n,k) + (k+1)S(2^n,k+1), \tag{1.33}$$
and using (1.31) we have
$$\nu_2((k+1)S(2^n,k+1)) = \nu_2(2^k/k!) > \nu_2(2^{k-1}/k!) = \nu_2(S(2^n,k)),$$
as claimed. Similar arguments can be used to obtain the value of $\nu_2(S(m,k))$ for values of m near a power of 2. For instance, $\nu_2(S(2^n+2,k+2)) = s_2(k) - 1$ if $\nu_2(k) \neq 0$ and $\nu_2(S(2^n+2,k+2)) = s_2(k+1) - 1$ if $\nu_2(k+1) > 1$.

In the general case, De Wannemacker [13] established the inequality

$$\nu_2\left(S(n,k)\right) \geq s_2(k) - s_2(n), \quad 0 \leq k \leq n. \tag{1.34}$$

Figure 6 shows the difference $\nu_2(S(n,k)) - s_2(k) + s_2(n)$ in the cases $k = 101$ and $k = 129$.

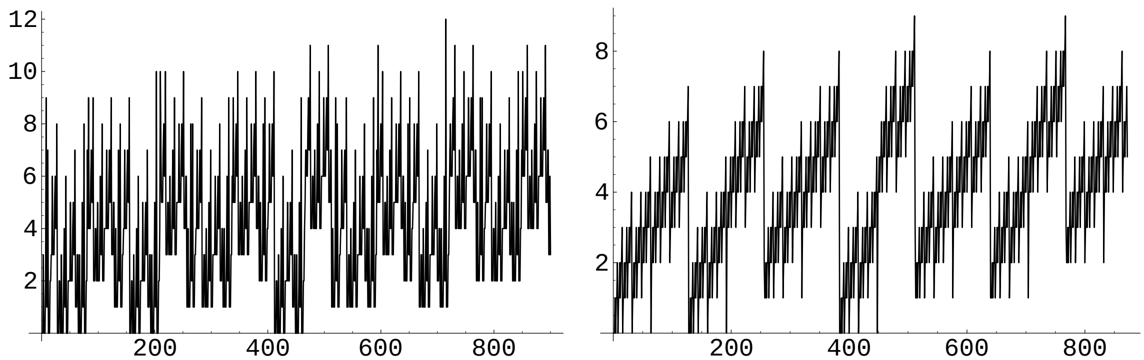

Figure 6: De Wannemacker difference for $k = 101$ and $k = 129$

2 The elementary cases

This section presents, for sake of completeness, the 2-adic valuation of $S(n,k)$ for $1 \leq k \leq 4$. The formulas for $S(n,k)$ come from (1.27). The arguments are all elementary.

Lemma 2.1. *The Stirling numbers of order* 1 *are given by* $S(n,1) = 1$*, for all* $n \in \mathbb{N}$*. Therefore*

$$\nu_2(S(n,1)) = 0. \tag{2.1}$$

Lemma 2.2. *The Stirling numbers of order* 2 *are given by* $S(n,2) = 2^n - 1$*, for all* $n \in \mathbb{N}$*. Therefore*

$$\nu_2(S(n,2)) = 0. \tag{2.2}$$

Lemma 2.3. *The Stirling numbers of order* 3 *are given by*

$$S(n,3) = \tfrac{1}{2}(3^{n-1} - 2^n + 1). \tag{2.3}$$

Moreover,

$$\nu_2(S(n,3)) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases} \tag{2.4}$$

Proof. Iterate the recurrence (1.29) to obtain

$$2^n - 1 = S(n,3) - \sum_{k=1}^{N-1} 3^k(2^{n-k}-1) - 3^N S(n-N,3), \tag{2.5}$$

and with $N = n-1$ we have

$$S(n,3) = 2^n - 1 - \sum_{k=1}^{n-2} 3^k(2^{n-k}-1). \tag{2.6}$$

If n is odd, then $S(n,3)$ is odd and $\nu_2(S(n,3)) = 0$.

For n even, the recurrence (1.29) yields

$$S(n,3) = 2^{n-1} + 3\cdot 2^{n-2} - 4 + 3^2 S(n-2,3). \tag{2.7}$$

As an inductive step, assume that $S(n-2,3) = 2T_{n-2}$, with T_{n-2} odd. Then (2.7) yields

$$\tfrac{1}{2}S(n,3) = 2^{n-2} + 3\cdot 2^{n-3} + 3^2 T_{n-2} - 2, \tag{2.8}$$

and we conclude that $S(n,3)/2$ is an odd integer. Therefore $\nu_2(S(n,3)) = 1$ as claimed. □

We now present a second proof of this result using elementary properties of the valuation ν_2. In particular, we use the ultrametric inequality

$$\nu_2(x_1+x_2) \geq \text{Min}\{\nu_2(x_1),\, \nu_2(x_2)\}. \tag{2.9}$$

The inequality is strict unless $\nu(x_1) = \nu_2(x_2)$. This inequality is equivalent to (1.6).

The powers of 3 modulo 8 satisfy $3^m + 1 \equiv 2 + (-1)^{m+1} \bmod 8$, because $3^{2k} \equiv 1 \bmod 8$. Therefore, $3^m + 1 = 8t + 3 + (-1)^{m+1}$ for some $t \in \mathbb{Z}$. Now

$$\nu_2(8t) = 3 + \nu_2(t) > \nu_2(3 + (-1)^{m+1}), \tag{2.10}$$

and the ultrametric inequality (2.9) yields

$$\nu_2(3^m+1) = \nu_2(3+(-1)^{m+1}) = \begin{cases} 2 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases} \tag{2.11}$$

Using $2S(n,3) = 3^{n-1} + 1 - 2^n$ and $\nu_2(2^n) = n > 2 \geq \nu_2(3^{n-1}+1)$, we conclude that

$$\nu_2(S(n,3)) = \nu_2(3^{n-1}+1-2^n) - 1 = \nu_2(3^{n-1}+1) - 1. \tag{2.12}$$

Lemma 2.3 now follows from (2.11).

We now discuss the Stirling number of order 4.

Lemma 2.4. *The Stirling numbers of order* 4 *are given by*

$$S(n,4) = \tfrac{1}{6}(4^{n-1} - 3^n - 3\cdot 2^{n+1} - 1). \tag{2.13}$$

Moreover,

$$\nu_2(S(n,4)) = \begin{cases} 1 & \textit{if } n \quad \textit{is odd,} \\ 0 & \textit{if } n \quad \textit{is even.} \end{cases} \tag{2.14}$$

Proof. The expression for $S(n, 4)$ comes from (1.27). To establish the formula for $\nu_2(S(n, 4))$, we use the recurrence (1.29) in the case $k = 4$:

$$S(n, 4) = S(n-1, 3) + 4S(n-1, 4). \tag{2.15}$$

For n even, the value $S(n-1, 3)$ is odd, so that $S(n, 4)$ is odd and $\nu_2(S(n, 4)) = 0$. For n odd, $S(n, 4)$ is even, since $S(n-1, 3)$ is even. Then (2.15), written as

$$\tfrac{1}{2}S(n, 4) = \tfrac{1}{2}S(n-1, 3) + 2S(n-1, 4), \tag{2.16}$$

and the value $\nu_2(S(n-1, 3)) = 1$, show that the right hand side of (2.16) is odd, yielding $\nu_2(S(n, 4)) = 1$. □

3 The Stirling numbers of order 5

The elementary cases discussed in the previous section are the only ones for which the 2-adic valuation $\nu_2(S(n, k))$ is easy to compute. The graph in figure 7 shows $\nu_2(S(n, 5))$.

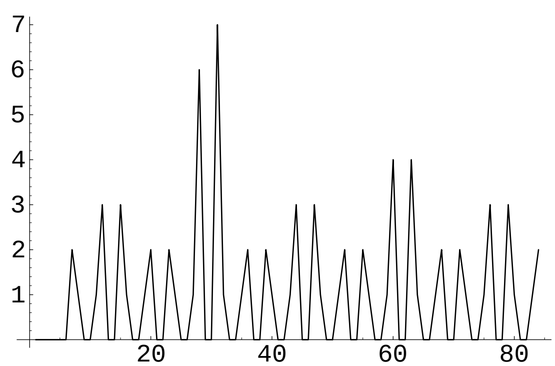

Figure 7: The 2-adic valuation of $S(n, 5)$

The explicit formula (1.27) yields

$$S(n, 5) = \tfrac{1}{24}(5^{n-1} - 4^n + 2 \cdot 3^n - 2^{n+1} + 1). \tag{3.1}$$

We now discuss the valuation $\nu_2(S(n, 5))$ in terms of the m-levels introduced in Section 1. The 1-level consists of two classes: $\{C_{1,0}, C_{1,1}\}$. None of these classes are constant, so we split them into $\{C_{2,0}, C_{2,1}, C_{2,2}, C_{2,3}\}$. The parity of $S(n, 5)$ determines two of them.

Lemma 3.1. *The Stirling numbers $S(n, 5)$ satisfy*

$$S(n, 5) \equiv \begin{cases} 1 \mod 2 & \text{if } n \equiv 1, \text{ or } 2 \mod 4, \\ 0 \mod 2 & \text{if } n \equiv 3, \text{ or } 0 \mod 4, \end{cases} \tag{3.2}$$

for all $n \in \mathbb{N}$.

Proof. The recurrence $S(n,5) = S(n-1,4) + 5S(n-1,5)$ and the parity

$$S(n,4) \equiv \begin{cases} 1 \mod 2 & \text{if } n \equiv 0 \mod 2, \\ 0 \mod 2 & \text{if } n \equiv 1 \mod 2, \end{cases} \tag{3.3}$$

give the result by induction. □

Corollary 3.2. *The 2-adic valuations of the Stirling numbers $S(n,5)$ satisfy*

$$\nu_2(S(4n+1,5)) = \nu_2(S(4n+2,5)) = 0 \textit{ for all } n \in \mathbb{N}. \tag{3.4}$$

The corollary states that the classes $C_{2,1}$ and $C_{2,2}$ are constant, so the $2-$level is

$$2 - \text{level}: \quad \{C_{2,0},\, C_{2,3}\}. \tag{3.5}$$

This confirms part of the main conjecture: here $m_0 = 3$ in view of $2^2 < 5 \leq 2^3$ and the first level where we find constant classes is $m_0 - 1 = 2$.

Remark. Corollary 3.2 reduces the discussion of $\nu_2(S(n,5))$ to the indices $n \equiv 0$ or $3 \mod 4$. These two branches can be treated in parallel. Introduce the notation

$$q_n := \nu_2(S(n,5)), \tag{3.6}$$

and consider the table of values

$$X := \{q_{4i},\, q_{4i+3} : i \geq 2\}. \tag{3.7}$$

This starts as

$$X = \{1,\, 1,\, 3,\, 3,\, 1,\, 1,\, 2,\, 2,\, 1,\, 1,\, 6,\, \mathbf{7},\, 1,\, 1,\, \ldots\}, \tag{3.8}$$

and after a while it continues as

$$X = \{\ldots, 1,\, 1,\, 2,\, 2,\, 1,\, 1,\, \mathbf{11},\, 6,\, 1,\, 1,\, 2,\, 2,\, \ldots\}. \tag{3.9}$$

We observe that $q_{4i} = q_{4i+3}$ for most indices.

Definition 3.3. The index i is called *exceptional* if $q_{4i} \neq q_{4i+3}$.

The first exceptional index is $i = 7$ where $q_{28} = 11 \neq q_{31} = 6$. The list of exceptional indices continues as $\{7,\, 39,\, 71,\, 103,\, \ldots\}$.

Conjecture 3.4. *The set of exceptional indices is $\{32j + 7 : j \geq 1\}$.*

We now consider the class

$$C_{2,0} := \{q_{4i} = \nu_2(S(4i), 5) : i \geq 2\}, \tag{3.10}$$

where we have omitted the first term $S(4,5) = 0$. The class $C_{2,0}$ starts as

$$C_{2,0} = \{1, 3, 1, 2, 1, 6, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, \ldots\}, \tag{3.11}$$

and it splits according to the parity of the index i into

$$C_{3,4} = \{q_{8i+4} : i \geq 1\} \text{ and } C_{3,0} = \{q_{8i} : i \geq 1\}. \tag{3.12}$$

It is easy to check that $C_{3,0}$ is constant.

Proposition 3.5. *The Stirling numbers of order* 5 *satisfy*

$$\nu_2(S(8i, 5)) = 1 \textit{ for all } i \geq 1. \tag{3.13}$$

Proof. We analyze the identity

$$24S(8i, 5) = 5^{8i-1} - 4^{8i} + 2 \cdot 3^{8i} - 2^{8i+1} + 1 \tag{3.14}$$

modulo 32. Using $5^8 \equiv 1$ and $5^7 \equiv 13$, we obtain $5^{8i-1} \equiv 13$. Also, $4^{8i} \equiv 2^{8i+1} \bmod 0$. Finally, $3^{8i} \equiv 81^{2i} \equiv 17^{2i} \equiv 1$. Therefore

$$5^{8i-1} - 4^{8i} + 2 \cdot 3^{8i} - 2^{8i+1} + 1 \equiv 16 \bmod 32. \tag{3.15}$$

We obtain that $24S(8i, 5) = 32t + 16$ for some $t \in \mathbb{N}$, and this yields $3S(8i, 5) = 2(2t+1)$. Therefore $\nu_2(S(8i, 5)) = 1$. □

We now consider the class $C_{3,4}$.

Proposition 3.6. *The Stirling numbers of order* 5 *satisfy*

$$\nu_2(S(8i + 4, 5)) \geq 2 \textit{ for all } i \geq 1. \tag{3.16}$$

Proof. We analyze the identity

$$24S(8i + 4, 5) = 5^{8i+3} - 4^{8i+4} + 2 \cdot 3^{8i+4} - 2^{8i+5} + 1 \tag{3.17}$$

modulo 32. Using $5^8 \equiv 1$, $5^3 \equiv 29$, $3^8 \equiv 1$, $3^4 \equiv 17$ and $2^4 \equiv 16$ modulo 32, we obtain

$$24S(8i + 4, 5) \equiv 0 \bmod 32. \tag{3.18}$$

Therefore $24S(8i + 4, 5) = 32t$ for some $t \in \mathbb{N}$, and this yields $\nu_2(S(8i + 4, 5) \geq 2$. □

Note. Lengyel [9] established that

$$\nu_2(k!S(n,k)) = k-1, \tag{3.19}$$

for $n = a2^q$, a odd and $q \geq k-2$. In the special case $k=5$, this yields $\nu_2(S(n,5)) = 1$ for $n = a2^q$ and $q \geq 3$. These values of n have the form $n = 8a \cdot 2^{q-3}$, so this is included in Proposition 3.5.

Remark. A similar argument yields

$$\nu_2(S(8i+3,5)) = 1 \text{ and } \nu_2(S(8i+7,5)) \geq 2. \tag{3.20}$$

We conclude that

$$3-\text{level}: \quad \{C_{3,4},\, C_{3,7}\}. \tag{3.21}$$

This illustrates the main conjecture: each of the classes of the 2-level produces a constant class and a second one in the 3-level.

We now consider the class $C_{3,4}$ and its splitting as $C_{4,4}$ and $C_{4,12}$. The data for $C_{3,4}$ starts as

$$C_{3,4} = \{3,\, 2,\, 6,\, 2,\, 3,\, 2,\, 4,\, 2,\, 3,\, 2,\, 5,\, 2,\, 3,\, 2,\, 4,\, 2,\, 3,\, 2,\, 11,\, 2,\, 3,\, 2, \ldots\}. \tag{3.22}$$

This suggests that the values with even index are all 2. This is verified below.

Proposition 3.7. *The Stirling numbers of order* 5 *satisfy*

$$\nu_2(S(16i+4,5)) = 2 \textit{ for all } \geq 1. \tag{3.23}$$

Proof. We analyze the identity

$$24S(16i+4,5) = 5^{16i+3} - 4^{16i+4} + 2 \cdot 3^{16i+4} - 2^{16i+5} + 1 \tag{3.24}$$

modulo 64. Using $5^{16} \equiv 1$, $5^3 \equiv 61$, $3^{16} \equiv 1$ and $3^4 \equiv 17$, we obtain

$$5^{16i+3} - 4^{16i+4} + 2 \cdot 3^{16i+4} - 2^{16i+5} + 1 \equiv 32 \bmod 64. \tag{3.25}$$

Therefore $24S(16i+4,5) = 64t+32$ for some $t \in \mathbb{N}$. This gives $3S(16i+4,5) = 4(2t+1)$, and it follows that $\nu_2(S(16i+4,5)) = 2$. □

Note. A similar argument shows that $\nu_2(S(16i+12,5)) \geq 3$, $\nu_2(S(16i+7,5)) = 2$ and $\nu_2(S(16i+15,5)) \geq 3$. Therefore the 4-level is $\{C_{4,12},\, C_{4,15}\}$.

This splitting process of the classes can be continued and, according to our main conjecture, the number of elements in the m-level is always constant. To prove the statement similar to Propositions 3.5 and 3.7, we must analyze the congruence

$$24S(2^m i + j, 5) \equiv 5^{2^m i + j - 1} - 4^{2^m i + j} + 2 \cdot 3^{2^m i + j} - 2^{2^m i + j + 1} + 1 \bmod 2^{m+2}. \tag{3.26}$$

We present a proof of this conjecture, for the special case $k=5$, in the next section.

Lundell [11] studied the Stirling-like numbers

$$T_p(n,k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n, \tag{3.27}$$

where the prime p is fixed, and the index j is omitted in the sum if it is divisible by p. Clarke [5] conjectured that

$$\nu_p(k!\, S(n,k)) = \nu_p(T(n,k)). \tag{3.28}$$

From this conjecture he derives an expression for $\nu_2(S(n,5))$ in terms of the zeros of the form $f_{0,5}(x) = 5 + 10 \cdot 3^x + 5^x$ in the ring of 2-adic integers $\mathbb{Z}_2$.

Theorem 3.8. *Let u_0 and u_1 be the 2-adic zeros of the function $f_{0,5}$. Then, under the assumption that conjecture (3.28) holds, we have*

$$\nu_2(S(n,5)) = \begin{cases} -1 + \nu_2(n - n_0) & \text{if } n \text{ is even,} \\ -1 + \nu_2(n - n_1) & \text{if } n \text{ is odd.} \end{cases} \tag{3.29}$$

Here u_0 is the unique zero of $f_{0,5}$ that satsifies $u_0 \in 2\mathbb{Z}_2$ and u_1 is the other zero of $f_{0,5}$ and satisfies $u_1 \in 1 + 2\mathbb{Z}_2$.

Clarke also obtained in [5] similar expressions for $\nu_2(S(n,6))$ and $\nu_2(S(n,7))$ in terms of zeros of the functions

$$f_{0,6} = -6 - 20 \cdot 3^x - 6 \cdot 5^x \text{ and } f_{0,7} = 7 + 35 \cdot 3^x + 21 \cdot 5^x + 7^x.$$

4 Proof of the main conjecture for $k = 5$

The goal of this section is to prove the main conjecture in the case $k = 5$. The parameter m_0 is 3 in view of $2^2 < 5 \leq 2^3$. In the previous section we have verified that $m_0 - 1 = 2$ is the first level for constant classes. We now prove this splitting of classes.

Theorem 4.1. *Assume $m \geq m_0$. Then the m-level consists of exactly two split classes: $C_{m,j}$ and $C_{m,j+2^{m-1}}$. They satisfy $\nu_2(C_{m,j}) > m-3$ and $\nu_2(C_{m,j+2^{m-1}}) > m-3$. Then exactly one, call it C^1, satisfies $\nu_2(C^1) = \{m-2\}$ and the other one, call it C^2, satisfies $\nu_2(C^2) > m-2$.*

The proof of this theorem requires several elementary results of 2-adic valuations.

Lemma 4.2. *For $m \in \mathbb{N}$, $\nu_2\left(5^{2^m} - 1\right) = m + 2$.*

Proof. Start at $m = 1$ with $\nu_2(24) = 3$. The inductive step uses

$$5^{2^{m+1}} - 1 = (5^{2^m} - 1) \cdot (5^{2^m} + 1).$$

Now $5^k + 1 \equiv 2 \bmod 4$ so that $5^{2^m} + 1 = 2\alpha_1$ with α_1 odd. Thus

$$\nu_2(5^{2^{m+1}} - 1) = \nu_2(5^{2^m} - 1) + \nu_2(5^{2^m} + 1) = (m+2) + 1 = m + 3.$$

□

The same type of argument produces the next lemma.

Lemma 4.3. *For* $m \in \mathbb{N}$, $\nu_2(3^{2^m} - 1) = m + 2$.

Lemma 4.4. *For* $m \in \mathbb{N}$, $\nu_2(5^{2^m} - 3^{2^m}) = m + 3$.

Proof. The inductive step uses

$$5^{2^{m+1}} - 3^{2^{m+1}} = (5^{2^m} - 3^{2^m}) \times \left((5^{2^m} - 1) + (3^{2^m} + 1) \right).$$

Therefore $\nu_2(5^{2^m} - 1) = m + 2$ and $3^{2^m} \equiv 1 \bmod 4$, thus $\nu_2(3^{2^m} + 1) = 1$. We conclude that

$$\nu_2((5^{2^m} - 1) + (3^{2^m} + 1)) = \text{Min}\{m + 2, 1\} = 1.$$

We obtain

$$\nu_2(5^{2^{m+1}} - 3^{2^{m+1}}) = m + 4, \tag{4.1}$$

and this concludes the inductive step. □

The recurrence (1.29) for the Stirling numbers $S(n, 5)$ is $S(n, 5) = 5S(n-1, 5) + S(n-1, 4)$. Iterating yields the next lemma.

Lemma 4.5. *Let* $t \in \mathbb{N}$. *Then*

$$S(n, 5) - 5^t S(n - t, 5) = \sum_{j=0}^{t-1} 5^j S(n - j - 1, 4). \tag{4.2}$$

Proof of theorem 4.1. We have already checked the conjecture for the 2-level. The inductive hypothesis states that there is an $(m-1)$-level survivor of the form

$$C_{m,k} = \{\nu_2(S(2^m n + k, 5)) : n \geq 1\}, \tag{4.3}$$

where $\nu_2(S(2^m n + k, 5)) > m - 2$. At the next level, $C_{m,k}$ splits into the two classes

$$\begin{aligned} C_{m+1,k} &= \{\nu_2(S(2^{m+1} n + k, 5)) : n \geq 1\} \quad \text{and} \\ C_{m+1,k+2^m} &= \{\nu_2(S(2^{m+1} n + k + 2^m, 5)) : n \geq 1\}, \end{aligned}$$

and every element of each of these two classes is greater or equal to $m - 1$. We now prove that one of these classes reduces to the singleton $\{m - 1\}$ and that every element in the other class is strictly greater than $m - 1$.

The first step is to use Lemma 4.5 to compare the values of $S(2^{m+1}n+k,5)$ and $S(2^{m+1}n+k+2^m,5)$. Define

$$M = 2^m - 1 \text{ and } N = 2^{m+1}n + k, \tag{4.4}$$

and use (1.29) to write

$$S(2^{m+1}n+k+2^m,5) - 5^{2^m} S(2^{m+1}n+k,5) = \sum_{j=0}^{M} 5^{M-j} S(N+j,4). \tag{4.5}$$

The next proposition establishes the 2-adic valuation of the right hand side.

Proposition 4.6. *With the notation as above,*

$$\nu_2\left(\sum_{j=0}^{M} 5^{M-j} S(N+j,4)\right) = m-1. \tag{4.6}$$

Proof. The explicit formula (1.27) yields $6S(n,4) = 4^{n-1} + 3\cdot 2^{n-1} - 3^n - 1$. Thus

$$\begin{aligned} 6\sum_{j=0}^{M} 5^{M-j} S(N+j,4) &= 4^{N-1}(5^{M+1}-4^{M+1}) + 2^{N-1}(5^{M+1}-2^{M+1}) \\ &\quad -3^N \times \tfrac{1}{2}(5^{M+1}-3^{M+1}) - \tfrac{1}{4}(5^{M+1}-1). \end{aligned}$$

The results in Lemmas 4.2, 4.3 and 4.4 yield

$$6\sum_{j=0}^{M} 5^{M-j} S(N+j,4) = 4^{N-1}\alpha_1 + 2^{N-1}\alpha_2 - 3^N\cdot 2^{m+2}\alpha_3 - 2^m\alpha_4, \tag{4.7}$$

with α_j odd integers. Write this as

$$6\sum_{j=0}^{M} 5^{M-j} S(N+j,4) = 2^{N-1}\left(2^{N-1}\alpha_1+\alpha_2\right) - 2^m\left(4\alpha_3 3^N + 1\right) \equiv T_1 + T_2.$$

Then $\nu_2(T_1) = N-1 > m = \nu_2(T_2)$, and we obtain

$$\nu_2\left(\sum_{j=0}^{M} 5^{M-j} S(N+j,4)\right) = m-1. \tag{4.8}$$

We conclude that

$$S(2^{m+1}n+k+2^m,5) - 5^{2^m} S(2^{m+1}n+k,5) = 2^{m-1}\alpha_5, \tag{4.9}$$

with α_5 odd. Define

$$X := 2^{-m+1}S(2^{m+1}n+k+2^m,5) \text{ and } Y := 2^{-m+1}S(2^{m+1}n+k,5). \tag{4.10}$$

Then X and Y are integers and $X - Y \equiv 1 \bmod 2$, so that they have opposite parity. If X is even and Y is odd, we obtain

$$\nu_2\left(S(2^{m+1}n+k+2^m,5)\right) > m-1 \text{ and } \nu_2\left(S(2^{m+1}n+k,5)\right) = m-1. \tag{4.11}$$

The case X odd and Y even is similar. This completes the proof. □

There are four classes at the first level corresponding to the residues modulo 4, two of which are constant. The complete determination of the valuation $\nu_2(S(n,5))$ is now determined by the choice of class when we move from level m to $m+1$. We consider only the branch starting at indices congruent to 0 modulo 4; the case of 3 modulo 4 is similar. Now there is single class per level that we write as

$$C_{m,j} = \{q_{2^m i+j} : i \in \mathbb{N}\}, \tag{4.12}$$

where $j = j(m)$ is the index that corresponds to the non-constant class at the m-level. The first few examples are listed below.

$$\begin{aligned}
C_{2,4} &= \{q_{4i+4} : i \in \mathbb{N}\} \\
C_{3,4} &= \{q_{8i+4} : i \in \mathbb{N}\} \\
C_{4,12} &= \{q_{16i-4} : i \in \mathbb{N}\} \\
C_{5,28} &= \{q_{32i-4} : i \in \mathbb{N}\} \\
C_{6,28} &= \{q_{64i-36} : i \in \mathbb{N}\} \\
C_{7,156} &= \{q_{128i-100} : i \in \mathbb{N}\} \\
C_{8,156} &= \{q_{256i-100} : i \in \mathbb{N}\} \\
C_{9,156} &= \{q_{512i-356} : i \in \mathbb{N}\} \\
C_{10,156} &= \{q_{1024i-868} : i \in \mathbb{N}\}
\end{aligned}$$

We have observed a connection between the indices $j(m)$ and the set of exceptional indices I_1 in (5.4).

Conjecture 4.7. *Construct a list of numbers $\{c_i : i \in \mathbb{N}\}$ according to the following rule: let $c_1 = 8$ (the first index in the class $C_{2,4}$), and then define c_j as the first value on $C_{m,j}$ that is strictly bigger than c_{j-1}. The set C begins as*

$$C = \{8, 12, 28, 60, 92, 156, 412, 668, 1180, \ldots\}. \tag{4.13}$$

Then, starting at 156, the number $c_i \in I_1$.

5 Some approximations

In this section we present some approximations to the function $\nu_2(S(n,5))$. These approximations were derived empirically, and they support our belief that 2-adic valuations of Stirling numbers can be well approximated by simple integer combinations of 2-adic valuations of integers.

For each prime p, define

$$\lambda_p(m) = \frac{1}{2}\left(1 - (-1)^{m \bmod p}\right). \tag{5.1}$$

First approximation. Define

$$f_1(m) := \lfloor \frac{m+1}{2} \rfloor + 112\lambda_2(m) + 50\lambda_2(m+1). \tag{5.2}$$

Then $\nu_2(S(m,5))$ and $\nu_2(f_1(m))$ agree for most values. The first time they differ is at $m = 156$ where

$$\nu_2(S(156,5)) - \nu_2(f_1(156)) = 4.$$

The first few indices for which $\nu_2(S(m,5)) \neq \nu_2(f_1(m))$ are $\{156, 287, 412, 668, 799, \ldots\}$.

Conjecture 5.1. *Define*

$$x_1(m) = 156 + 125\lfloor \frac{4m}{3} \rfloor + 6\lfloor \frac{2m+1}{3} \rfloor \tag{5.3}$$

and

$$I_1 = \{x_1(m) : \; m \geq 0\}. \tag{5.4}$$

Then $\nu_2(S(m,5)) = \nu_2(f_1(m))$ *unless* $m \in I_1$.

The parity of the exceptions in I_1 is easy to establish: every third element is odd and the even indices of I_1 are on the arithmetic progression $256m + 156$.

Second approximation. We now consider the error

$$Err_1(m,5) := \nu_2(S(m,5)) - \nu_2(f_1(m)). \tag{5.5}$$

Observe that $Err_1(m,5) = 0$ for m outside I_1.

Define

$$\begin{aligned} m_3(m) &:= (m+2) \mod 3, \\ \alpha_m &:= \lambda_3(m+2)\,(1 + \lambda_3(m)) + \lambda_2(m+1)\lambda_3(m), \end{aligned}$$

and

$$f_2(m) = \binom{2m_3}{m_3} \lfloor \frac{m+2}{3} \rfloor + 208\lambda_3(m+1) + 27\lambda_2(m)\lambda_3(m). \tag{5.6}$$

The next conjecture improves the prediction of Conjecture 5.1.

Conjecture 5.2. *Consider the set* $I_2 = \{x_2(m) : \; m \geq 0\}$, *where*

$$x_2(m) = 109 + 107\lfloor \frac{4m+2}{3} \rfloor + 85\lfloor \frac{4m+1}{3} \rfloor. \tag{5.7}$$

Then

$$Err_1(x_1(m),5) = (-1)^{\alpha_m} \nu_2(f_2(m)), \tag{5.8}$$

unless $m \in I_2$.

We now present one final improvement. Define

$$Err_2(m,5) := Err_1(x_1(m),5) - (-1)^{\alpha_m} \nu_2 f_2(m)). \tag{5.9}$$

Define $\beta(m) = \alpha_m + (-1)^{m+1}\lambda_3(m)$ and

$$f_3(m) = 4^{1-\lambda_3(m)} \lfloor \frac{m+2}{3} \rfloor + \lambda_3(m)\,(85\lambda_3(m) + 8\lambda_2(m+1) + 2\lambda_3(m+1))\,. \tag{5.10}$$

Conjecture 5.3. $Err_2(x_2(m),5)$ *agrees with* $(-1)^{\beta(x_2(m))}\, \nu_2(f_3(x_2(m)))$ *for most values of* $m \in \mathbb{N}$.

6 A sample of pictures

In this section we present data that illustrate the wide variety of behavior for the 2-adic valuation of Stirling numbers $S(n, k)$. Several features are common to all. For instance we observe the appearance of an *empty region* from below the graph. The graph of $\nu_2(S(n, 126))$ shows the basic function $\nu_2(n)$ in the interior of the graph. The dark objects in $\nu_2(S(n, 195))$ and $\nu_2(S(n, 260))$ correspond to an oscillation between two consecutive values. We are lacking an explanation of these features.

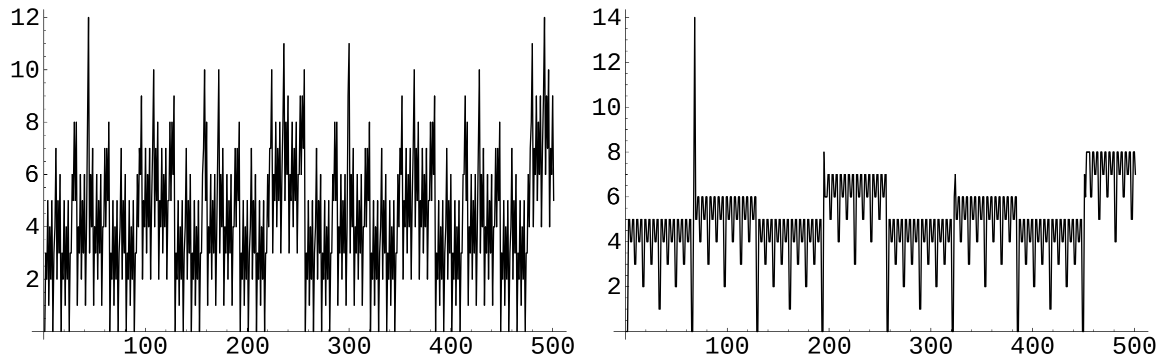

Figure 8: The data for $S(n, 80)$ and $S(n, 126)$.

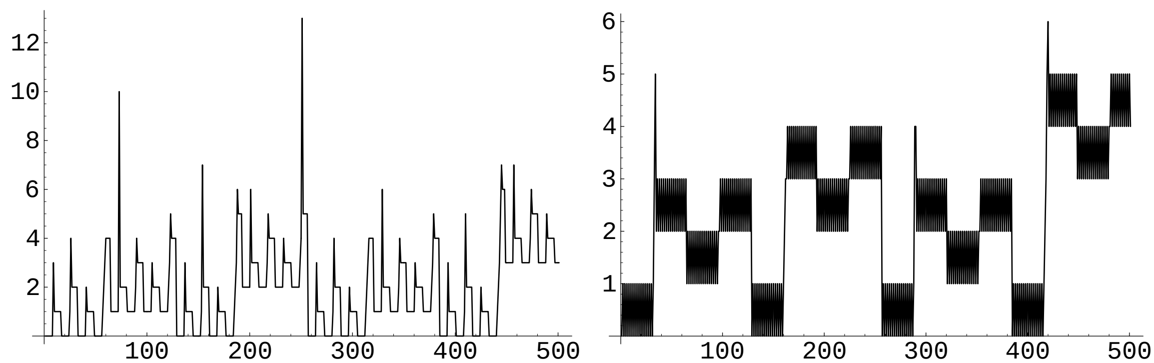

Figure 9: The data for $S(n, 146)$ and $S(n, 195)$.

7 Conclusions

We have presented a conjecture that describes the 2-adic valuation of the Stirling numbers $S(n, k)$. This conjecture is established for $k = 5$.

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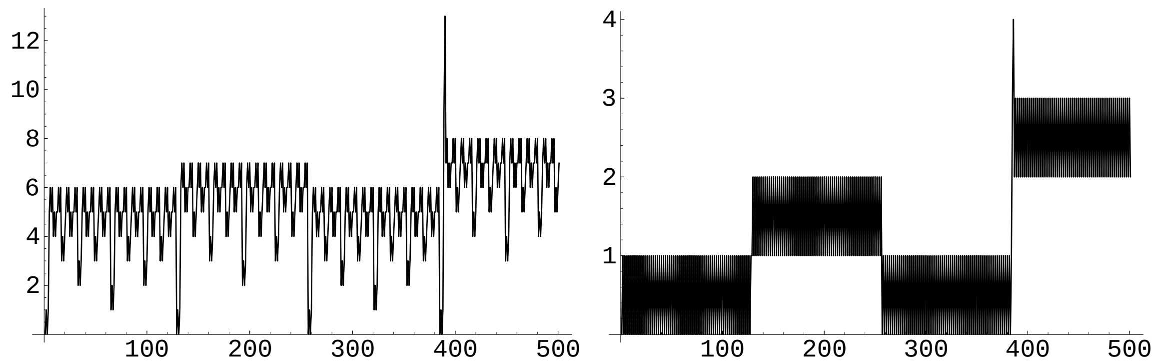

Figure 10: The data for $S(n, 252)$ and $S(n, 260)$.

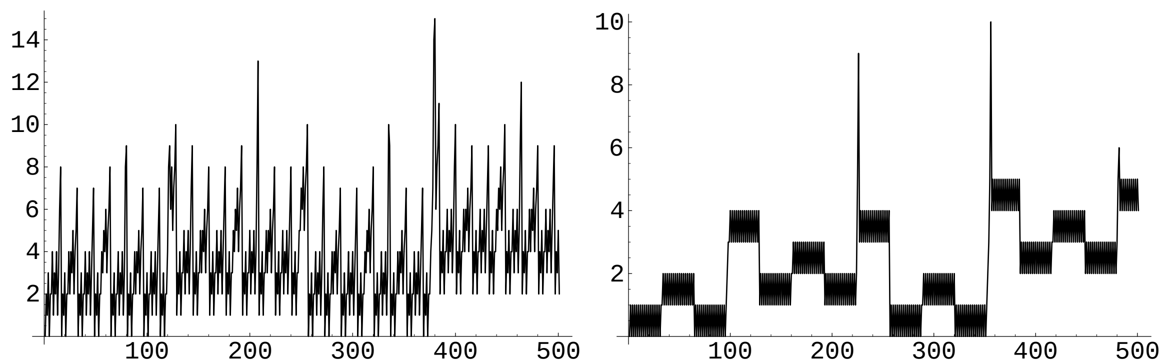

Figure 11: The data for $S(n, 279)$ and $S(n, 324)$.

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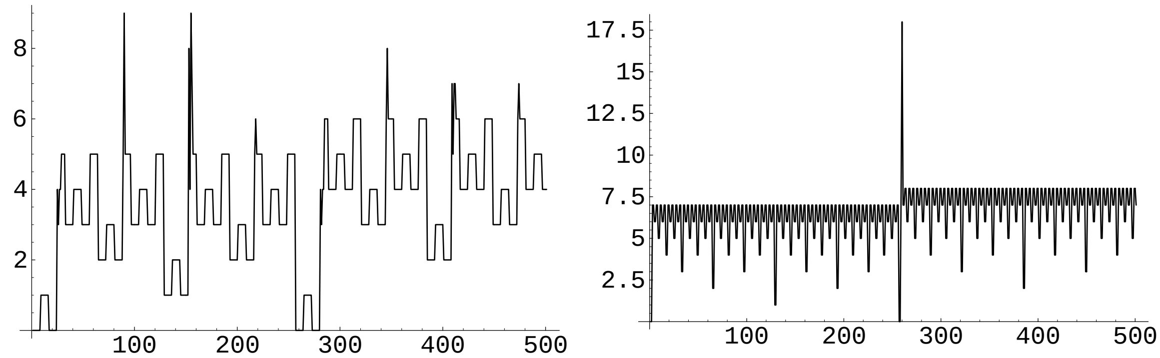

Figure 12: The data for $S(n, 465)$ and $S(n, 510)$.

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