CONVERGENCE OF A PLANAR TRANSFORMATION

In [1, 2] we have developed a theory of the *rational Landen transformations*. These are transformations on the parameters of a rational function that preserve its integral. This is the rational analogue of the *elliptic Landen transformation*:

$$a_1 = (a+b)/2 \text{ and } b_1 = \sqrt{ab}$$

that preserve the elliptic integral

$$G(a,b) \quad = \quad \int_0^\pi \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}.$$

A general theory of convergence has been discussed in [3] and the specific case of degree 6 appears in [1]. The proof of convergence of these transformations employs their connection to integrals.

Problem. Prove directly that the transformation

$$\begin{aligned} a_1 &\to \frac{a_1a_2 + 5a_1 + 5a_2 + 9}{(a_1+a_2+2)^{4/3}} \\ a_2 &= \frac{a_1+a_2+6}{(a_1+a_2+2)^{2/3}} \end{aligned}$$

converges provided $R_-(a_1^0, a_2^0) > 0$. The function R_- is the branch of the resolvent curve

$$R(a_1,a_2) \quad = \quad 4a_1^3 + 4a_2^3 - 18a_1a_2 - a_1^2a_2^2 + 27 = 0$$

that intersects the third quadrant.

The transformation above, coupled with

$$\begin{aligned} b_0 &\to \frac{b_0+b_1+b_2}{a_1+a_2+2)^{2/3}} \\ b_1 &\to \frac{b_0(a_2+3) + 2b_1 + b_2(a_1+3)}{a_1+a_2+2} \\ b_2 &\to \frac{b_0+b_2}{(a_1+a_2+2)^{1/3}} \end{aligned}$$

preserve the integral

$$U \quad = \quad \int_0^\infty \frac{b_0x^4 + b_1x^2 + b_2}{x^6 + a_1x^4 + a_2x^2 + 1}\, dx.$$

References

[1] Boros, G. - Moll, V.: *A rational Landen transformation. The case of degree six.* Contemporary Mathematics **251**, 83-91. Analysis, Geometry, Number Theory: The Mathematics of Leon Ehrenpreis; E.L. Grinberg, S. Berhanu, M. Knopp, G. Mendoza and E.T. Quinto editors.

[2] Boros, G. - Moll, V.: *Landen transformations and the integration of rational functions.* Math. Comp. **71** (2002), 649-668.

[3] Hubbard, J. - Moll, V.: *A geometric view of the rational Landen transformation.* Preprint.