LOGCONCAVITY OF A SEQUENCE OF COEFFICIENTS ARISING FROM A FAMILY OF DEFINITE INTEGRALS

A finite sequence of real numbers $\{d_0, d_1, \cdots, d_m\}$ is said to be *unimodal* if there exists an index $0 \leq j \leq m$ such that $d_0 \leq d_1 \leq \cdots \leq d_j$ and $d_j \geq d_{j+1} \geq \cdots \geq d_m$. The sequence $\{d_0, d_1, \cdots, d_m\}$ with $d_j \geq 0$ is said to be *logarithmically concave* (or *log concave* for short) if $d_{j+1}d_{j-1} \leq d_j^2$ for $1 \leq j \leq m-1$.

The question deals with

$$P_m(a) = \sum_{l=0}^{m} d_l(m) a^l,$$

with

$$d_l(m) = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}.$$

The polynomials $P_m(a)$ arise in the formula

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a).$$

We have established the following result.

Theorem. If P is a polynomial with positive nondecreasing coefficients, then $P(x+1)$ is unimodal.

This is used in [1] to establish the unimodality of $d_l(m)$ and other sequences.

The question is to establish the logconcavity of the coefficients $d_l(m)$. We believe that much more is true. For a sequence $\mathbf{a} = \{a_i : 1 \leq i \leq n\}$ define the operator

$$\mathfrak{L}(\mathbf{a}) := \{a_j^2 - a_{j+1}a_{j-1} : 1 \leq j \leq n\}$$

We say that $\mathbf{a}$ is k-logconcave if $\mathfrak{L}^r(\mathbf{a})$ is a positive sequence for $1 \leq r \leq k$. In this language, logconcavity is 1-logconcave. A sequence is called ∞-logconcave if it is k-logconcave for every k.

Problem 1. Prove that the coefficients $d_l(m)$ form an ∞-logconcave sequence.

It is perhaps simpler to establish this for the binomial coefficients.

Problem 2. Prove that the binomial coefficients form an ∞-logconcave sequence.

References

[1] Boros, G. - Moll, V.: *A criterion for unimodality.* Elec. Jour. Combinatorics, **6**, 1999, #R10.
[2] Wilf, H.: *generatingfunctionology.* Academic Press, 1990.