# THE EXPANSION OF THE TRIPLE SQUARE ROOT

In [1] we have shown that the Taylor expansion of $h(c) = \sqrt{a+\sqrt{1+c}}$ is given by

$$\sqrt{a+\sqrt{1+c}} \quad = \quad \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} N_{0,4}(a;k-1)c^k.$$

Here

$$N_{0,4}(a;m) \quad = \quad \int_0^{\infty} \frac{dx}{(x^4+2ax^2+1)^{m+1}}.$$

The integrals can be expressed as

$$N_{0,4}(a;m) \quad = \quad \frac{\pi}{2^{m+3/2}\,(a+1)^{m+1/2}} P_m(a)$$

where the polynomial $P_m(a)$ is given by

$$P_m(a) \quad = \quad 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k.$$

The special cases $c=1$ and $a=c^2$ appear in BROMWICH [2], page 192, exercise 21, and in [3] 1.114.1.

Based on symbolic experiments we propose a formula for the Taylor series expansion of the **triple square root**

$$h_{a,b}(c) \quad := \quad \sqrt{a+\sqrt{b+\sqrt{1+c}}}.$$

Prove that the coefficients of the Taylor series expansion

$$h_{a,b}(c) \quad = \quad \sum_{n=0}^{\infty} \beta_n(a,b)c^n$$

are given by

$$\beta_0(a,b) \quad = \quad \sqrt{a+\sqrt{1+b}}$$

and

$$\beta_n(a,b) \quad = \quad \frac{(-1)^{n-1}}{n\,2^{2n+1}} \sum_{k=0}^{n-1} \binom{2n-2-k}{n-1} q^{-k-1/2} P_k^*(a,\sqrt{1+b}),$$

where $q := (1+b)(a+\sqrt{1+b})$ and

$$P_k^*(a,z) \quad = \quad z^k P_k(a/z)$$

is the homogenization of $P_k$.

**Note**. The presence of the homogeneous polynomials $P_k^*(a,z)$ suggests a geometric interpretation of $h_{a,b}(c)$.

## References

[1] Boros, G. - Moll, V.: *The double square root, Jacobi polynomials and Ramanujan's master theorem.* Journal of Computational and Applied Mathematics **130** (2001), 337-344.

[2] Bromwich, T.J.: *An Introdcution to the Theory of Infinite Series*, 2nd. Edition, MacMillan, New York, 1926.

[3] Gradshteyn, I.S. and Rhyzik, I.M.: *Tables of Integrals, Series and Products*, 6-th edition, Academic Press, New York, 2000. Edited by Alan Jeffrey and Daniel Zwillinger.