# THE $p$-ADIC VALUATION OF A SEQUENCE OF COEFFICIENTS

The question deals with

$$P_m(a) \quad = \quad \sum_{l=0}^{m} d_l(m) a^l,$$

with

$$d_l(m) \quad = \quad 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}.$$

The polynomials $P_m(a)$ arise in the formula

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \quad = \quad \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a).$$

We are interested in the divisibility properties of the coefficients $d_l(m)$. The function $S_p(m)$, for $p$ prime, is the sum of the digits of $m$ when written in base $p$. The $p$-adic valuation of an integer $n$ is defined as the exact exponent of $p$ that divides $n$.

We have established the following result:

**Theorem**. The 2-adic valuation of the constant term $d_0(m)$ is given by

$$\nu_2(d_0(m)) \quad = \quad S_2(m) - 2m.$$

The 2-adic valuation of the linear term $d_1(m)$ is given by

$$\nu_2(d_1(m)) \quad = \quad 1 - 2m + \nu_2\left(\binom{m+1}{2}\right) + S_2(m).$$

**Problem 1**. Produce similar formulas for the other coefficients $\nu_2(d_j(m))$.

The case of the prime 3 seems more difficult.

**Problem 2**. Prove the existence of a sequence of positive integers $m_j$ such that $\nu_3(d_1(m_j) = 0$. Extensive calculations show that

$$q_j := m_{j+1} - m_j \quad \in \quad \{2, 7, 20, 61, 182, \cdots\}$$

where the sequence $\{q_j\}$ above is defined by $q_1 = 2$ and $q_{j+1} = 3q_j + (-1)^{j+1}$. Establish similar results for other odd primes.

## References


[1] Boros, G. - Moll, V., Shallit, J.: *The 2-adic valuation of the coefficients of a polynomial.* Revista Scientia, to appear.