TWO FAMILIES OF POLYNOMIALS WITH ZEROS ON A VERTICAL LINE

The polynomial

$$P_m(a) \quad = \quad \sum_{l=0}^{m} d_l(m) a^l,$$

with

$$d_l(m) \quad = \quad 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}$$

appears in the formula

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \quad = \quad \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a).$$

This can be found in [1].

The number of terms in the sum giving $d_l(m)$ is $m+1-l$. In [2] we have shown the existence of two families of polynomials $\{\alpha_l(m),\ \beta_l(m)\}$ such that

$$d_l(m) \quad = \quad \frac{1}{l!\, m!\, 2^{m+l}} \left(\alpha_l(m) \prod_{k=1}^{m} (4k-1) - \beta_l(m) \prod_{k=1}^{m} (4k+1) \right).$$

The first few are

$$\begin{aligned} \alpha_0(m) &= 1 & \beta_0(m) &= 0 \\ \alpha_1(m) &= 2m+1 & \beta_1(m) &= 1 \\ \alpha_2(m) &= 2(2m^2+2m+1) & \beta_2(m) &= 2(2m+1) \\ \alpha_3(m) &= 4(2m+1)(m^2+m+3) & \beta_3(m) &= 12(m^2+m+1) \end{aligned}$$

Problem 1. Prove that the coefficients of $\alpha_l(m)$ and $\beta_l(m)$ are positive integers and that the degrees of α_l and β_l are l and $l-1$, respectively.

Problem 2. Prove that all the zeros of $\alpha_l(m)$ and $\beta_l(m)$ lie on the line $\text{Re}(m) = -1/2$.

Problem 3. Study the limit of α_l and β_l as $l \to \infty$. Perhaps this would produce entire functions with zeros on a fixed vertical line.

References

[1] Boros, G. - Moll, V.: *An integral hidden in Gradshteyn and Rhyzik.* Journal of Computational and Applied Mathematics **106**, 1999, 361-368.

[2] Boros, G. - Moll, V., Shallit, J.: *The 2-adic valuation of the coefficients of a polynomial.* Revista Scientia, to appear.