

THE 2-ADIC VALUATION OF A SEQUENCE ARISING FROM A RATIONAL INTEGRAL

TEWODROS AMDEBERHAN, DANTE MANNA, AND VICTOR H. MOLL

ABSTRACT. We analyze properties of the 2-adic valuation of an integer sequence that originates from an explicit evaluation of a quartic integral. We also give a combinatorial interpretation of the valuations of this sequence.

1. INTRODUCTION

Wallis's formula

$$(1.1) \quad \int_0^\infty \frac{dx}{(x^2 + 1)^{m+1}} = \frac{\pi}{2^{2m+1}} \binom{2m}{m}$$

is one of the earlier instances of evaluation of definite integrals where the result contains interesting arithmetical and combinatorial properties. In this paper we examine such connection for the integral

$$(1.2) \quad N_{0,4}(a; m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}.$$

The condition $a > -1$ is imposed for convergence. The evaluation

$$(1.3) \quad N_{0,4}(a, m) = \frac{\pi}{2} \frac{P_m(a)}{[2(a+1)]^{m+\frac{1}{2}}}$$

where

$$(1.4) \quad P_m(a) = \sum_{l=0}^m d_l(m) a^l$$

with

$$(1.5) \quad d_l(m) = 2^{-2m} \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}, \quad 0 \leq l \leq m,$$

appeared in [4]. The reader will find in [2] a survey of the different proofs of (1.3) and an introduction to the many issues involved in the evaluation of definite integrals in [8].

The study of combinatorial aspects of the sequence $d_l(m)$ was initiated in [3] where the authors show that $d_l(m)$ form a *unimodal* sequence, that is, there exists and index l^* such that $d_0(m) \leq \dots \leq d_{l^*}(m)$ and $d_{l^*}(m) \geq \dots \geq d_m(m)$. The fact that $d_l(m)$ satisfies the stronger condition of *logconcavity* $d_{l-1}(m)d_{l+1}(m) \leq d_l^2(m)$

Date: November 20, 2007.

1991 Mathematics Subject Classification. Primary 11B50, Secondary 05A15.

Key words and phrases. valuations, compositions, generating functions.

has been recently established in [6]. We consider here arithmetical properties of the sequence $d_{l,m}$. It is more convenient to analyze the auxiliary sequence

$$(1.6) \quad A_{l,m} = \frac{l! m!}{2^{m+l}} d_{l,m} = \frac{l! m!}{2^{m-l}} \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}$$

for $m \in \mathbb{N}$ and $0 \leq l \leq m$. The integral (1.2) is then given explicitly as

$$(1.7) \quad N_{0,4}(a; m) = \frac{\pi}{\sqrt{2} m! (4(2a+1))^{m+1/2}} \sum_{l=0}^m A_{l,m} \frac{a^l}{l!}.$$

In [5] it is shown that $A_{l,m} \in \mathbb{N}$. Observe that the computation of $A_{l,m}$ using (1.6) is more efficient if l is close to m . For instance,

$$(1.8) \quad A_{m,m} = 2^m (2m)! \text{ and } A_{m-1,m} = 2^{m-1} (2m-1)! (2m+1).$$

A second method to compute $A_{l,m}$, efficient now when l is small, has been discussed in [5]. There, it is shown that $A_{l,m}$ is a linear combination (with polynomial coefficients) of

$$(1.9) \quad \prod_{k=1}^m (4k-1) \text{ and } \prod_{k=1}^m (4k+1).$$

For example,

$$(1.10) \quad A_{0,m} = \prod_{k=1}^m (4k-1) \text{ and } A_{1,m} = (2m+1) \prod_{k=1}^m (4k-1) - \prod_{k=1}^m (4k+1).$$

The results described in this paper started with some empirical observations on the behavior of the 2-adic valuation of $A_{l,m}$, i.e. $\nu_2(A_{l,m})$. Recall that, for $x \in \mathbb{N}$, the 2-adic valuation $\nu_2(x)$ is the highest power of 2 that divides x . This is extended to $x = a/b \in \mathbb{Q}$ via $\nu_2(x) = \nu_2(a) - \nu_2(b)$. From (1.10) it follows that $A_{0,m}$ is odd, so $\nu_2(A_{0,m}) = 0$. Moreover,

$$(1.11) \quad \nu_2(A_{1,m}) = \nu_2(m(m+1)) + 1,$$

i.e., the main result of [5]. We present as Theorem 2.1, an expression for $\nu_2(A_{l,m})$ that generalizes (1.11).

The study of the sequence

$$(1.12) \quad X(l) := \{\nu_2(A_{l,l+m-1}) : m \geq 1\}$$

requires the introduction of two operators, F and T , defined in (4.1) and (4.2), respectively. The iteration of these operators creates an integer vector

$$(1.13) \quad \Omega(l) := \{n_1, n_2, n_3, \dots, n_{\omega(l)}\}, \text{ with } n_i \in \mathbb{N},$$

associated to the index $l \in \mathbb{N}$. We call $\Omega(l)$ the *reduction sequence* of l . See (4.2) for the precise definition of the integers n_j . The structure of $X(l)$ motivates the following definition.

Definition 1.1. Let $s \in \mathbb{N}$, $s \geq 2$. We say that a sequence $\{a_j : j \in \mathbb{N}\}$ is *simple of length s* (or *s -simple*) if s is the largest integer such that for each $t \in \{0, 1, 2, \dots\}$, we have

$$(1.14) \quad a_{st+1} = a_{st+2} = \dots = a_{s(t+1)}.$$

The sequence $\{a_j : j \in \mathbb{N}\}$ is said to have a *block structure* if it is s -simple for some $s \geq 2$.

Section 2 presents two proofs of the expression for $\nu_2(A_{l,m})$. Section 3 shows that $X(l)$ is a simple sequence of length $2^{1+\nu_2(l)}$. In Section 4 an algorithm generating the vector $\Omega(l)$ is described in detail. A combinatorial interpretation of $\Omega(l)$, as the composition of l , is provided in Section 5. Theorem 5.5 gives $\Omega(l)$ in terms of the dyadic expansion of l . More precisely, if $\{k_1, \dots, k_n : 0 \leq k_1 < k_2 < \dots < k_n\}$ is the unique collection of distinct nonnegative integers such that $l = \sum_{i=1}^n 2^{k_i}$, then the reduction sequence $\Omega(l)$ of l is $\{k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\}$. Finally, the last section contains a conjecture on symmetries of the graph of $\nu_2(A_{l,m})$.

2. THE 2-ADIC VALUATION OF $A_{l,m}$

In this section we prove that $\nu_2(A_{l,m})$ agrees with $\nu_2((m+1-l)_{2l}) + l$. The first proof actually produces the latter term in a natural way starting from the former. The second proof employs the WZ-machinery [9] to prove the identity (2.1).

Theorem 2.1. *The 2-adic valuation of $A_{l,m}$ satisfies*

$$(2.1) \quad \nu_2(A_{l,m}) = \nu_2((m+1-l)_{2l}) + l,$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol for $k \geq 1$. For $k = 0$, we define $(a)_0 = 1$.

Proof. First proof. We have

$$(2.2) \quad \nu_2(A_{l,m}) = l + \nu_2\left(\sum_{k=l}^m T_{m,k} \frac{(m+k)!}{(m-k)!(k-l)!}\right),$$

where

$$(2.3) \quad T_{m,k} = \frac{(2m-2k)!}{2^{m-k}(m-k)!}.$$

The identity

$$(2.4) \quad T_{m,k} = \frac{(2(m-k))!}{2^{m-k}(m-k)!} = (2m-2k-1)(2m-2k-3)\cdots 3 \cdot 1$$

shows that $T_{m,k}$ is an odd integer. Then (2.2) can be written as

$$\begin{aligned} \nu_2(A_{l,m}) &= l + \nu_2\left(\sum_{k=0}^{m-l} T_{m,l+k} \frac{(m+k+l)!}{(m-k-l)!k!}\right) \\ &= l + \nu_2\left(\sum_{k=0}^{m-l} T_{m,l+k} \frac{(m-k-l+1)_{2k+2l}}{k!}\right). \end{aligned}$$

The term corresponding to $k = 0$ is singled out as we write

$$\nu_2(A_{l,m}) = l + \nu_2\left(T_{m,l}(m-l+1)_{2l} + \sum_{k=1}^{m-l} T_{m,l+k} \frac{(m-k-l+1)_{2k+2l}}{k!}\right).$$

The claim

$$(2.5) \quad \nu_2\left(\frac{(m-k-l+1)_{2k+2l}}{k!}\right) > \nu_2((m-l+1)_{2l})$$

for any k , $1 \leq k \leq m-l$, will complete the proof.

To prove (2.5) we use the identity

$$\frac{(m-k-l+1)_{2k+2l}}{k!} = (m-l+1)_{2l} \cdot \frac{(m-l-k+1)_k (m+l+1)_k}{k!}$$

and the fact that the product of k consecutive numbers is always divisible by $k!$. This follows from the identity

$$(2.6) \quad \frac{(a)_k}{k!} = \binom{a+k-1}{k}.$$

Now if $m+l$ is odd,

$$(2.7) \quad \nu_2\left(\frac{(m-l-k+1)_k}{k!}\right) \geq 0 \text{ and } \nu_2((m+l+1)_k) > 0,$$

and if $m+l$ is even

$$(2.8) \quad \nu_2\left(\frac{(m+l+1)_k}{k!}\right) \geq 0 \text{ and } \nu_2((m-l-k+1)_k) > 0.$$

This proves (2.5) and establishes the theorem.

Second proof. Define the numbers

$$(2.9) \quad B_{l,m} := \frac{A_{l,m}}{2^l(m+1-l)_{2l}}.$$

We need to prove that $B_{l,m}$ is odd. The WZ-method [9] provides the recurrence

$$B_{l-1,m} = (2m+1)B_{l,m} - (m-l)(m+l+1)B_{l+1,m}, \quad 1 \leq l \leq m-1.$$

Since the initial values $B_{m,m} = 1$ and $B_{m-1,m} = 2m+1$ are odd, it follows that $B_{l,m}$ is an odd integer. \square

3. PROPERTIES OF THE FUNCTION $\nu_2(A_{l,m})$

Let $l \in \mathbb{N} \cup \{0\}$ be fixed. In this section we describe properties of the function $\nu_2(A_{l,m})$. In particular, we show that each of these sequences has a block structure.

Theorem 3.1. *Let $l \in \mathbb{N} \cup \{0\}$ be fixed. Then for $m \geq l$, we have*

$$(3.1) \quad \nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) = \nu_2(m+l+1) - \nu_2(m-l+1).$$

Proof. From (2.1) and $(a)_k = (a+k-1)!/(a-1)!$, we have

$$(3.2) \quad \nu_2(A_{l,m}) = \nu_2\left(\frac{(m+l)!}{(m-l)!}\right) + l.$$

This implies

$$\begin{aligned} \nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) &= \nu_2\left(\frac{(m+l+1)!}{(m-l+1)!}\right) - \nu_2\left(\frac{(m+l)!}{(m-l)!}\right) \\ &= \nu_2\left(\frac{(m+l+1)! (m-l)!}{(m-l+1)! (m+l)!}\right) \\ &= \nu_2(m+l+1) - \nu_2(m-l+1). \end{aligned}$$

\square

The next corollary is a special case of Theorem 3.1.

Corollary 3.2. *The sequence $\nu_2(A_{l,m})$ satisfies*

- 1) $\nu_2(A_{l,l+1}) = \nu_2(A_{l,l})$.
- 2) For l even,

$$\nu_2(A_{l,l+3}) = \nu_2(A_{l,l+2}) = \nu_2(A_{l,l+1}) = \nu_2(A_{l,l}).$$

- 3) *The sequence $\nu_2(A_{1,m})$ is 2-simple; i.e., $\nu_2(A_{1,m+1}) = \nu_2(A_{1,m})$ for m odd. In fact,*

$$A_{1,m} = \{2, 2, 3, 3, 2, 2, 4, 4, 2, 2, \dots\}.$$

Fix $k, l \in \mathbb{N}$ and let $\mu := 1 + \nu_2(l)$. Define the following sets

$$(3.3) \quad C_{k,l} := \{l + k \cdot 2^\mu + j : 0 \leq j \leq 2^\mu - 1\},$$

which will be instrumental in proving the main result of this section; i.e., $\{\nu_2(A_{l,m})\}$ is $2^{1+\nu_2(l)}$ -simple.

We begin by showing that these sets form a partition of \mathbb{N} . Moreover, for fixed $k, l \in \mathbb{N}$ the set $C_{k,l}$ has cardinality 2^μ and the 2-adic valuation of $\{A_{l,m} : m \in C_{k,l}\}$ is constant. For example, if $l \in \mathbb{N}$ is odd, then $\mu = 1$ and

$$(3.4) \quad C_{k,l} = \{l + 2k, l + 2k + 1\}.$$

The next result is immediate.

Lemma 3.3. *Let $l \in \mathbb{N}$ be fixed. The sets $\{C_{k,l} : k \geq 0\}$ form a disjoint partition of \mathbb{N} ; namely,*

$$(3.5) \quad \{m \in \mathbb{N} : m \geq l\} = \bigcup_{k \geq 0} C_{k,l},$$

and $C_{r,l} \cap C_{t,l} = \emptyset$, whenever $r \neq t$.

Lemma 3.4. *Fix $l \in \mathbb{N}$ and let $\mu = \nu_2(2l)$.*

- 1) *The sequence $\{\nu_2(A_{l,m}) : m \in C_{k,l}\}$ is constant. We denote this value by $\nu_2(C_{k,l})$.*
- 2) *For $k \geq 0$, $\nu_2(C_{k+1,l}) \neq \nu_2(C_{k,l})$.*

Proof. Suppose $0 \leq j \leq 2^\mu - 2$. Since $\nu_2(2l) = \mu \leq \nu_2(k \cdot 2^\mu)$, then

$$(3.6) \quad \nu_2(2l + k \cdot 2^\mu) \geq \nu_2(2l) = \mu > \nu_2(j + 1),$$

because $j + 1 < 2^\mu$. Therefore

$$(3.7) \quad \nu_2(2l + k \cdot 2^\mu + j + 1) = \nu_2(j + 1) = \nu_2(k \cdot 2^\mu + j + 1).$$

Using these facts and (3.1), we obtain

$$\begin{aligned} \nu_2(A_{l,l+k \cdot 2^\mu + j + 1}) - \nu_2(A_{l,l+k \cdot 2^\mu + j}) &= \nu_2(2l + k \cdot 2^\mu + j + 1) - \nu_2(k \cdot 2^\mu + j + 1) \\ &= \nu_2(j + 1) - \nu_2(j + 1) = 0 \end{aligned}$$

for consecutive values in $C_{k,l}$. This proves part 1). To prove part 2), it suffices to take elements $l + k \cdot 2^\mu + 2^\mu - 1 \in C_{k,l}$ and $l + (k + 1) \cdot 2^\mu \in C_{k+1,l}$ and compare their 2-adic values. Again by (3.1), we have

$$\begin{aligned} \nu_2(A_{l,l+(k+1) \cdot 2^\mu}) - \nu_2(A_{l,l+(k+1) \cdot 2^\mu - 1}) &= \nu_2(2l + (k + 1) \cdot 2^\mu) - \nu_2((k + 1) \cdot 2^\mu) \\ &= \mu + \nu_2(2l \cdot 2^{-\mu} + k + 1) - \mu - \nu_2(k + 1) \\ &= \nu_2(2l \cdot 2^{-\mu} + k + 1) - \nu_2(k + 1) \neq 0. \end{aligned}$$

The last step follows from $2l \cdot 2^{-\mu}$ being odd and thus $2l \cdot 2^{-\mu} + k + 1$ and $k + 1$ having opposite parities. This completes the proof. \square

Theorem 3.5. *For each $l \geq 1$, the set $\{\nu_2(A_{l,m}) : m \geq l\}$ is an s -simple sequence, with $s = 2^{1+\nu_2(l)}$.*

Proof. From Lemma 3.3 and Lemma 3.4, we know that $\nu_2(\cdot)$ maintains a constant value on each of the disjoint sets $C_{k,l}$. The length of each of these blocks is $2^{1+\nu_2(l)}$. \square

4. THE ALGORITHM AND ITS COMBINATORIAL INTERPRETATION

In this section we describe an algorithm that extracts from the sequence $X(1) := \{\nu_2(A_{1,m}) : m \geq 1\}$ its combinatorial information. We begin with the definition of the operators F and T mentioned in the Introduction.

Definition 4.1. The maps F and T . These are defined by

$$(4.1) \quad F(\{a_1, a_2, a_3, \dots\}) := \{a_1, a_1, a_2, a_3, \dots\},$$

and

$$(4.2) \quad T(\{a_1, a_2, a_3, \dots\}) := \{a_1, a_3, a_5, a_7, \dots\}.$$

We employ the notation

$$(4.3) \quad c := \{\nu_2(m) : m \geq 1\} = \{0, 1, 0, 2, 0, 1, 0, 3, 0, \dots\}.$$

The algorithm:

- 1) Start with the sequence $X(l) := \{\nu_2(A_{l,l+m-1}) : m \geq 1\}$.
- 2) Find $n \in \mathbb{N}$ so that the sequence $X(l)$ is 2^n -simple. Define $Y(l) := T^n(X(l))$. At the initial stage, Theorem 3.5 ensures that $n = 1 + \nu_2(l)$.
- 3) Introduce the shift $Z(l) := Y(l) - c$.
- 4) Define $W(l) := F(Z(l))$.

If $W(l)$ is a constant sequence, then STOP; otherwise go to step 2) with W instead of X . Define $X_k(l)$ as the new sequence at the end of the $(k-1)$ th cycle of this process, with $X_1(l) = X(l)$.

Section 5 contains the justification for the steps of this algorithm. In particular, we prove that the sequences $X_k(l)$ have a block structure, so they can be used back in step 1 after each cycle. Theorem 5.3 states that the algorithm finishes in a finite number of steps and that $W(l)$ is essentially $X(j)$, for some $j < l$.

Definition 4.2. Let $\omega(l)$ be the number of cycles required for the algorithm to yield a constant sequence and denote by n_j the integers appearing in Step 2 of the algorithm. The integer vector

$$(4.4) \quad \Omega(l) := \{n_1, n_2, n_3, \dots, n_{\omega(l)}\}$$

is called the *reduction sequence* of l . The number $\omega(l)$ will be called the *reduction length* of l . The constant sequence obtained after $\omega(l)$ cycles is called the *reduced constant*.

TABLE 1. Reduction sequence for $1 \leq l \leq 15$.

l	binary form	$\Omega(l)$
4	100	3
5	101	1, 2
6	110	2, 1
7	111	1, 1, 1
8	1000	4
9	1001	1, 3
10	1010	2, 2
11	1011	1, 1, 2
12	1100	3, 1
13	1101	1, 2, 1
14	1110	2, 1, 1
15	1111	1, 1, 1, 1

In Corollary 5.8 we enumerate $\omega(l)$ as the number of ones in the binary expansion of l . Therefore the algorithm yields a constant sequence in a finite number of steps. In fact, the algorithm terminates after $O(\log_2(l))$ cycles as will follow directly from Corollary 5.8. Table 1 shows the results of the algorithm for $4 \leq l \leq 15$.

We now provide a combinatorial interpretation of $\Omega(l)$. This requires the composition of the index l .

Definition 4.3. Let $l \in \mathbb{N}$. The *composition* of l , denoted by $\Omega_1(l)$, is defined as follows: write l in binary form. Read the sequence from right to left. The first part of $\Omega_1(l)$ is the number of digits up to and including the first 1 read in the corresponding binary sequence; the second one is the number of additional digits up to and including the second 1 read, and so on.

Example 4.4. Reading off the values from Table 1, we obtain $\Omega_1(13) = \{1, 2, 1\}$ and $\Omega_1(14) = \{2, 1, 1\}$. Therefore $\Omega_1(13) = \Omega(13)$ and $\Omega_1(14) = \Omega(14)$. Corollary 5.6 shows that this is always true.

The next result describes the formation of $\Omega_1(l)$ from $\Omega_1(\lfloor l/2 \rfloor)$.

Lemma 4.5. *Given the values of $\Omega_1(l)$ for $2^j \leq l \leq 2^{j+1} - 1$, the list for $2^{j+1} \leq l \leq 2^{j+2} - 1$ is formed according to the following rule:*

l is even: add 1 to the first part of $\Omega_1(l/2)$ to obtain $\Omega_1(l)$;

l is odd: prepend a 1 to $\Omega_1(\frac{l-1}{2})$ to obtain $\Omega_1(l)$.

Proof. Let $x_1x_2 \cdots x_t$ be the binary representation of l . Then $x_1x_2 \cdots x_t0$ corresponds to $2l$. Thus, the first part of $\Omega_1(2l)$ is increased by 1, due to the extra 0 on the right. The relative position of the remaining 1s stays the same. A similar argument takes care of $\Omega_1(2l + 1)$. The extra 1 that is placed at the end of the binary representation gives the first 1 in $\Omega_1(2l + 1)$. \square

We now relate the 2-adic valuation of $A_{l,m}$ to that of $A_{\lfloor l/2 \rfloor, m}$.

Proposition 4.6. *Let*

$$(4.5) \quad \lambda_l := \frac{1 - (-1)^l}{2}, \quad M_0 := \lfloor \frac{m + \lambda_l}{2} \rfloor.$$

Then

$$(4.6) \quad \nu_2(A_{l,m}) = 2l - \lfloor l/2 \rfloor + \lambda_l \nu_2(M_0 - \lfloor l/2 \rfloor) + \nu_2(A_{\lfloor l/2 \rfloor, M_0}).$$

Proof. We present the details for $\nu_2(A_{2l,2m})$. Theorem 2.1 gives

$$\begin{aligned} \nu_2(A_{2l,2m}) &= \nu_2((2m - 2l + 1)_{4l}) + 2l \\ &= \nu_2((2m - 2l + 1)(2m - 2l + 2) \cdots (2m + 2l - 1)(2m + 2l)) + 2l \\ &= \nu_2(2^{2l}(m - l + 1)(m - l + 2) \cdots (m + l)) + 2l \\ &= 4l + \nu_2((m - l + 1)_{2l}) \\ &= 3l + \nu_2(A_{l,m}). \end{aligned}$$

A similar calculation shows that

$$(4.7) \quad \nu_2(A_{2l+1,2m}) = 3l + 2 + \nu_2(A_{l,m}) + \nu_2(m - l).$$

The general case then follows from Theorem 3.1. \square

Corollary 4.7. *The 2-adic valuation of $A_{l,m}$ satisfies*

$$(4.8) \quad \nu_2(A_{l,m}) = 2l + \nu_2(l!) + \sum_{k \geq 0} \lambda_{\lfloor l/2^k \rfloor} \nu_2(M_k - \lfloor l/2^{k+1} \rfloor)$$

where

$$(4.9) \quad M_k = \lfloor \frac{m + \lambda_l + 2\lambda_{\lfloor l/2 \rfloor} + \cdots + 2^k \lambda_{\lfloor l/2^k \rfloor}}{2^{1+k}} \rfloor = \lfloor \frac{m + \sum_{n=0}^k 2^n \lambda_{\lfloor l/2^n \rfloor}}{2^{1+k}} \rfloor.$$

Proof. This is a repeated application of Proposition 4.6. The first term results from

$$\begin{aligned} \sum_{k \geq 0} \left(2 \lfloor \frac{l}{2^k} \rfloor - \lfloor \frac{l}{2^{k+1}} \rfloor \right) &= 2l + \sum_{k \geq 1} \lfloor \frac{l}{2^k} \rfloor \\ &= 2l + \nu_2(l!). \end{aligned}$$

\square

5. VERIFICATION OF THE ALGORITHM AND THE REDUCTION SEQUENCE

In this section we show that the algorithm presented in Section 4 terminates after a finite numbers of cycles. Moreover, we prove that $\Omega(l)$, the reduction sequence of l , is identical to the composition sequence of l .

Notation: The constant sequences will be denoted by $(t) = \{t, t, t, \dots\}$.

Definition 5.1. A sequence $(a) = \{a_1, a_2, a_3, \dots\}$ is a *translate* of $(b) = \{b_1, b_2, b_3, \dots\}$ if $(a) = (b) + (t)$, for some constant sequence (t) . Addition of sequences is performed term by term.

We first consider the base case $l = 1$.

Lemma 5.2. *The initial case $l = 1$ satisfies*

$$(5.1) \quad W(1) = F(T(X(1)) - c) = (2),$$

where (c) is given in (4.3).

Proof. Since $\nu_2(A_{1,m}) = \nu_2(m(m+1)) + 1$ and $\nu_2(2m-1) = 0$, we have

$$T(X(1)) = \{\nu_2((2m-1)(2m)) + 1 : m \geq 1\} = \{\nu_2(m) + 2 : m \geq 1\} = c + (2).$$

Then the assertion follows from $F((t)) = (t)$ for a constant (t) . \square

Theorem 5.3. *The algorithm terminates after finitely many iterations. Furthermore, in each cycle, $W(l)$ is a translate of $X(j)$, for some $j < l$.*

Proof. Start by rewriting the terms in $X(l)$ as

$$\nu_2 \left(\frac{(m-1+2l)!}{(m-1)!} \right) + l = \nu_2((m-1+2l)(m-2+2l) \cdots (m+1)m) + l, \quad m \geq 1.$$

Then, the operator T acts on these to yield (for $m \geq 1$)

$$(5.2) \quad \begin{aligned} & \nu_2((2m-2+2l)(2m-3+2l) \cdots (2m)(2m-1)) + l \\ &= \nu_2((m-1+l) \cdots (m)) + 2l \\ &= \nu_2 \left(\frac{(m-1+l)!}{(m-1)!} \right) + 2l. \end{aligned}$$

Case I: l is even. From (5.2), we can easily obtain the relation

$$T(X(l)) = \left\{ \nu_2 \left(\frac{(m-1+l)!}{(m-1)!} \right) + l/2 + t : m \geq 1 \right\} = X(l/2) + (t), \quad t = 3l/2.$$

Case II: l is odd. Upon subtracting the sequence $c = \{\nu_2(m) : m \geq 1\}$ from (5.2) we get that

$$\nu_2 \left(\frac{(m+l-1)!}{m!} \right) + 2l = \nu_2 \left(\frac{(m+l-1)!}{m!} \right) + \frac{l-1}{2} + \frac{3(l-1)}{2} + 2,$$

for $m \geq 1$. Then, apply the operator F to the last sequence and find

$$W(l) = \left\{ \nu_2 \left(\frac{(m-2+l)!}{(m-1)!} \right) + \frac{l-1}{2} + t : m \geq 1 \right\} = X \left(\frac{l-1}{2} \right) + (t), \quad t = (3l+1)/2.$$

Here, we have utilized the property that $\nu_2(r!) = \nu_2((r-1)!)$, when $r \geq 1$ is odd. This justifies that the first term augmented in the sequence, as a result of the action of F , coincides with the next term (these are values at $m = 1$ and $m = 2$, respectively).

We can now conclude that in either of the two cases (or a combination thereof), the index l shrinks dyadically. Thus the reduction algorithm must end in a finite step into a translate of $X(1)$. Since Lemma 5.2 handles $X(1)$, the proof is completed. \square

Corollary 5.4. *For general $k \in \mathbb{N}$, the sequence $X_k(l)$ is 2^{n_k} -simple for some $n_k \in \mathbb{N}$.*

Theorem 5.5. *Let $\{k_1, \dots, k_n : 0 \leq k_1 < k_2 < \dots < k_n\}$, be the unique collection of distinct nonnegative integers such that*

$$(5.3) \quad l = \sum_{i=1}^n 2^{k_i}.$$

Then the reduction sequence $\Omega(l)$ of l is $\{k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\}$.

Proof. The argument of the proof is to check that the rules of formation for $\Omega_1(l)$ also hold for the reduction sequence $\Omega(l)$. The proof is divided according to the parity of l . The case l odd starts with $l = 1$, where the block length is 2. From Theorem 2.1 we obtain a constant sequence after iterating the algorithm once. Thus the algorithm terminates and the reduction sequence for $l = 1$ is $\Omega(1) = \{1\}$.

Now consider the general even case: $X(2l)$. Theorem 5.3 shows that applying T to this sequence yields a translate of $X(l)$. This does not affect the reduction sequence $\Omega(l)$, but the doubling of block length increases the first term of $\Omega(l)$ by 1. Therefore

$$(5.4) \quad \Omega(2l) = \{k_1 + 2, k_2 - k_1, \dots, k_n - k_{n-1}\}.$$

This is precisely what happens to the binary digits of l : if

$$l = \sum_{i=1}^n 2^{k_i}, \text{ then } 2l = \sum_{i=1}^n 2^{k_i+1}.$$

This concludes the argument for even indices.

For the general odd case, $X(2l + 1)$, we apply T , subtract c and then apply F . Again, by Theorem 5.3, this gives us a translate of $X(l)$. We conclude that, if the reduction sequence of l is

$$(5.5) \quad \{k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\},$$

then that of $2l + 1$ is

$$(5.6) \quad \{1, k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\}.$$

This is precisely the behavior of Ω_1 . The proof is complete. \square

Corollary 5.6. *The reduction sequence $\Omega(l)$ associated to an integer l is the sequence of compositions of l , that is,*

$$(5.7) \quad \Omega(l) = \Omega_1(l).$$

Corollary 5.7. *The reduced constant is $2l + \nu_2(l!) = \nu_2(A_{l,l})$.*

Proof. In Corollary 4.7, subtract the last term as per the reduction algorithm. \square

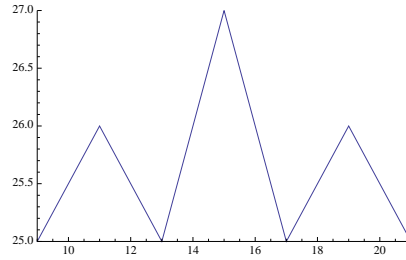
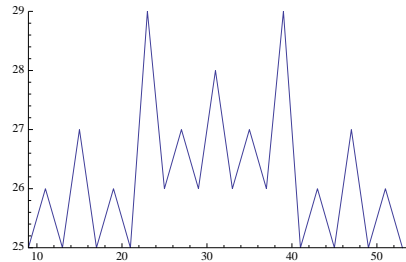
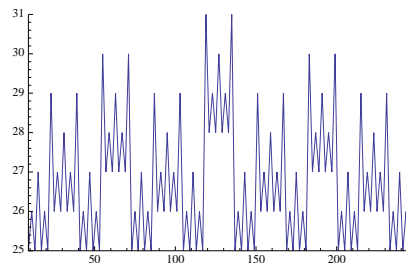
Corollary 5.8. *The set $\Omega(l)$ has cardinality*

$$(5.8) \quad s_2(l) = \text{the number of ones in the binary expansion of } l.$$

Note. The function $s_2(l)$ defined in (5.8) has recently appeared in a different divisibility problem. Lengyel [7] conjectured, and De Wannemacker [10] proved, that the 2-adic valuation of the Stirling numbers of the second kind $S(n, k)$ is given by

$$(5.9) \quad \nu_2(S(2^n, k)) = s_2(k) - 1.$$

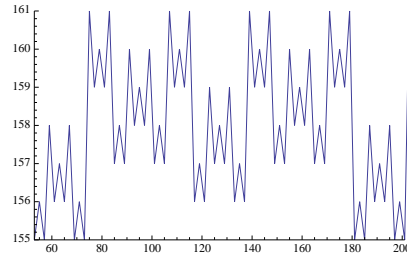
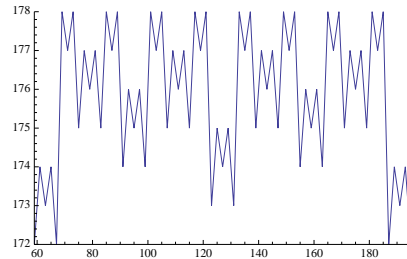
The reader will find in [1] a general study of the 2-adic valuation of Stirling numbers.

FIGURE 2. The beginning for $l = 9$ FIGURE 3. The continuation of $l = 9$ FIGURE 4. The pattern for $l = 9$ persists

Acknowledgements. The last author acknowledges the partial support of NSF-DMS 0409968. The second author was partially supported as a graduate student by the same grant. The authors wish to thank Aaron Jaggard for identifying their data with the composition sequence.

REFERENCES

- [1] T. Amdeberhan, D. Manna, and V. Moll. The 2-adic valuation of Stirling numbers. *Experimental Mathematics*, to appear in 2008.
- [2] T. Amdeberhan and V. Moll. A formula for a quartic integral: a survey of old proofs and some new ones. *Ramanujan Journal*, to appear in 2008.
- [3] G. Boros and V. Moll. A criterion for unimodality. *Elec. Jour. Comb.*, 6:1–6, 1999.

FIGURE 5. The initial pattern for $l = 53$ FIGURE 6. The initial pattern for $l = 59$

- [4] G. Boros and V. Moll. An integral hidden in Gradshteyn and Ryzhik. *Jour. Comp. Applied Math.*, 106:361–368, 1999.
- [5] G. Boros, V. Moll, and J. Shallit. The 2-adic valuation of the coefficients of a polynomial. *Scientia, Series A*, 7:37–50, 2001.
- [6] M. Kauers and P. Paule. A computer proof of Moll’s log-concavity conjecture. *Proc. Amer. Math. Soc.*, 135:3837–3846, 2007.
- [7] T. Lengyel. On the divisibility by 2 of the Stirling numbers of the second kind. *Fib. Quart.*, 32:194–201, 1994.
- [8] V. Moll. The evaluation of integrals: a personal story. *Notices of the AMS*, 49:311–317, 2002.
- [9] M. Petkovsek, H. Wilf, and D. Zeilberger. *A=B*. A. K. Peters, Ltd., 1st edition, 1996.
- [10] S. De Wannemacker. On the 2-adic orders of Stirling numbers of the second kind. *INTEGERS*, 5(1):A–21, 2005.

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
E-mail address: tamdeberhan@math.tulane.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA, B3H 3J5
E-mail address: dmanna@mathstat.dal.ca

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
E-mail address: vhm@math.tulane.edu