

ASYMPTOTIC VALUATIONS OF SEQUENCES SATISFYING FIRST ORDER RECURRENCES

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ABSTRACT. Let t_n be a sequence that satisfies a first order homogeneous recurrence $t_n = Q(n)t_{n-1}$, where $Q \in \mathbb{Z}[n]$. The asymptotic behavior of the p -adic valuation of t_n is described under the assumption that all the roots of Q in $\mathbb{Z}/p\mathbb{Z}$ have nonvanishing derivative.

1. INTRODUCTION

The p -adic valuation $\nu_p(x)$, for $x \in \mathbb{Q}$, $x \neq 0$, is defined by

$$(1.1) \quad x = p^{\nu_p(x)} \frac{a}{b},$$

where $a, b \in \mathbb{Z}$ and p divides neither a nor b . The value $\nu_p(0)$ is left undefined.

In this paper we establish the asymptotic behavior of the p -adic valuation of sequences that satisfy first order recurrences

$$(1.2) \quad t_n = Q(n)t_{n-1}, \quad n \geq 1,$$

where Q is a polynomial with integer coefficients. Among all the positive integer zeros of Q , let v be the maximum modulus. Take $n_0 > v$. Then the recurrence (1.2) is started at this index n_0 . This ensures the non-vanishing of t_n . Without loss of generality, we always assume $n_0 = 0$ and $t_0 = 1$. We also adopt the notation $t_n(Q)$ while referring to the sequence defined by (1.2).

The identity

$$(1.3) \quad \nu_p(t_n(Q)) = \sum_{i=1}^n \nu_p(Q(i)),$$

shows that only the zeros of Q in $\mathbb{Z}/p\mathbb{Z}$ contribute to the value of $\nu_p(t_n(Q))$. The main tool of our asymptotic analysis will be Hensel's lemma. The version stated here is reproduced from [4]:

Lemma 1.1 (Hensel). *Let $f(x) \in \mathbb{Z}_p[x]$ be a polynomial with coefficients in the p -adic integers \mathbb{Z}_p . Write $f'(x)$ for its formal derivative. If $f(x) \equiv$*

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0 mod p has a solution a_1 satisfying $f'(a_1) \not\equiv 0 \pmod{p}$, then there is a unique p -adic integer a such that $f(a) = 0$ and $a \equiv a_1 \pmod{p}$.

We now state our main theorem. This result is an asymptotic description of the valuation of the sequence t_n , defined by (1.2).

Theorem 1.2. *Let $Q \in \mathbb{Z}[n]$. Assume each of the roots of Q satisfies the hypothesis of Hensel's lemma. Let z_p denote the number of roots of Q in $\mathbb{Z}/p\mathbb{Z}$, that is,*

$$(1.4) \quad z_p := |\{b \in \{1, 2, \dots, p\} : Q(b) \equiv 0 \pmod{p}\}|.$$

Then the sequence $\{t_n\}$, defined in (1.2), obeys the estimate

$$(1.5) \quad \nu_p(t_n) \sim \frac{z_p n}{p-1} \text{ as } n \rightarrow \infty.$$

Motivation. The most elementary example is $Q(x) = x$. Theorem 1.2 yields $\nu_p(n!) \sim n/(p-1)$. This follows from the classical formula of Legendre

$$(1.6) \quad \nu_p(n!) = \frac{n - s_p(n)}{p-1},$$

where $s_p(n)$ is the sum of the digits of n in base p .

Our motivation for Theorem 1.2 comes from the study of the sequence $\{x_n\}$ defined by

$$(1.7) \quad x_n = \tan \sum_{k=1}^n \tan^{-1} k, \quad n \geq 1.$$

This same sequence satisfies the recursive relation

$$(1.8) \quad x_n = \frac{x_{n-1} + n}{1 - nx_{n-1}},$$

with initial condition $x_1 = 1$. The first few values are $\{1, -3, 0, 4, -\frac{9}{19}\}$, and in [2] it was conjectured that $x_n \neq 0$ for $n \geq 4$. Later this was proved in [1] using the 2-adic valuation of x_n . The sequence $\{x_n\}$ was linked in [1] to

$$(1.9) \quad \omega_n := (1 + 1^2)(1 + 2^2)(1 + 3^2) \cdots (1 + n^2),$$

which can be condensated as

$$(1.10) \quad \omega_n = (1 + n^2)\omega_{n-1}.$$

This corresponds to $Q(x) = x^2 + 1$ and it fits into the type of recurrences considered here.

Section 2 contains the proof of Theorem 1.2 and Section 3 presents examples illustrating the main result. In the last section we propose some future directions.

2. THE PROOF

In the proof we assume that Q has no roots in $\mathbb{N} \cup \{0\}$. The general situation can be reduced to this one by a shift of the independent variable.

The conclusion of Theorem 1.2 is trivial if $z_p = 0$, so we assume $z_p > 0$. Denote by b_1, b_2, \dots, b_{z_p} the zeros of Q in $\mathbb{Z}/p\mathbb{Z}$. The definition of t_n yields

$$(2.1) \quad \nu_p(t_n) = \sum_{i=1}^n \nu_p(Q(i)).$$

All sums below are assumed to run from $i = 1$ to n .

Only the indices congruent to b_j modulo p contribute to (2.1), thus

$$(2.2) \quad \nu_p(t_n) = \sum_{i \equiv b_1 \pmod{p}} \nu_p(Q(i)) + \dots + \sum_{i \equiv b_{z_p} \pmod{p}} \nu_p(Q(i))$$

where $1 \leq i \leq n$. For fixed $j \in \{1, 2, \dots, z_p\}$, we consider the term

$$(2.3) \quad \sum_{i \equiv b_j \pmod{p}} \nu_p(Q(i)).$$

Hensel's lemma produces a p -adic integer

$$(2.4) \quad \beta_j = \beta_{j,0} + \beta_{j,1}p + \dots + \beta_{j,k}p^k + \dots$$

such that $\beta_{j,k} \in \{0, 1, \dots, p-1\}$, $\beta_{j,0} \equiv b_j \pmod{p}$ and $Q(\beta_j) = 0$. Observe that if the representation (2.4) were finite, then β_j would be a non-negative integer root of Q . This possibility has been excluded. Introduce the notation

$$(2.5) \quad \gamma_{j,s} := \beta_{j,0} + p\beta_{j,1} + p^2\beta_{j,2} + \dots + p^s\beta_{j,s}p^s.$$

Definition 2.1. For $n \in \mathbb{N}$, let

$$(2.6) \quad r_n = \text{Max} \{j : p^j \text{ divides some } Q(i) \text{ for } 1 \leq i \leq n\}.$$

Lemma 2.2. The sequence $r_n \rightarrow \infty$ as $n \rightarrow \infty$. Moreover, for large n , we have $p^{r_n} \leq n^{\deg(Q)+1}$, hence $r_n = O(\log n)$.

Proof. Hensel's lemma shows that $\gamma_{j,s}$ satisfies $Q(\gamma_{j,s}) \equiv 0 \pmod{p^{s+1}}$. For any given $M > 0$, choose an integer $s > M$. Taking $n > \gamma_{j,s-1}$ we have that $i := \gamma_{j,s-1} \in \{1, 2, \dots, n\}$ and $p^s | Q(i)$. The definition of r_n implies that $r_n \geq s > M$. Therefore $r_n \rightarrow \infty$ as $n \rightarrow \infty$. Now observe that p^{r_n} divides $|Q(i)|$ for some $1 \leq i \leq n$. The estimate

$$(2.7) \quad p^{r_n} \leq |Q(i)| \leq \text{Max}\{|Q(1)|, \dots, |Q(n)|\} \leq Cn^{\deg(Q)}$$

gives the upper bound on r_n . The constant C depends only on the coefficients of Q . \square

Now

$$\sum_{i \equiv b_j \pmod{p}} \nu_p(Q(i)) = \sum_{i \equiv \gamma_{j,0} \pmod{p}} 1 + \sum_{i \equiv \gamma_{j,1} \pmod{p^2}} 1 + \dots + \sum_{i \equiv \gamma_{j,r_n-1} \pmod{p^{r_n}}} 1,$$

where all sums range over $1 \leq i \leq n$. The bound

$$(2.8) \quad \left\lfloor \frac{n}{p^s} \right\rfloor \leq \sum_{i \equiv \gamma_{j,s} \pmod{p}} 1 \leq \left\lfloor \frac{n}{p^s} \right\rfloor + 1$$

yields

$$\begin{aligned} \sum_{i \equiv b_j \pmod{p}} \nu_p(Q(i)) &\geq \left(\frac{n}{p} - 1 \right) + \left(\frac{n}{p^2} - 1 \right) + \cdots + \left(\frac{n}{p^{r_n}} - 1 \right) \\ &= n \left(\frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{r_n}} \right) - r_n \\ &= \frac{n}{p-1} (1 - p^{-r_n}) - r_n. \end{aligned}$$

Therefore

$$\frac{p-1}{n} \sum_{i \equiv b_j \pmod{p}} \nu_p(Q(i)) \geq 1 - p^{-r_n} - \frac{(p-1)r_n}{n}$$

and passing to the limit we conclude that

$$(2.9) \quad \liminf_{n \rightarrow \infty} \frac{p-1}{n} \sum_{i \equiv b_j \pmod{p}} \nu_p(Q(i)) \geq 1.$$

Similarly, using the upper bound in (2.8) we obtain

$$\sum_{i \equiv b_j \pmod{p}} \nu_p(Q(i)) \leq r_n + \frac{n}{p-1},$$

and it follows that

$$(2.10) \quad \limsup_{n \rightarrow \infty} \frac{p-1}{n} \sum_{i \equiv b_j \pmod{p}} \nu_p(Q(i)) \leq 1.$$

Therefore, Theorem 1.2 has been established.

3. EXAMPLES

In this section we present some examples illustrating Theorem 1.2.

Definition 3.1. *Given a polynomial Q and a prime p , we say that $a \in \mathbb{Z}/p\mathbb{Z}$ is a Hensel zero of Q if $Q(a) \equiv 0 \pmod{p}$ and $Q'(a) \not\equiv 0 \pmod{p}$. The prime p is called a Hensel prime for Q if all the zeros of Q in $\mathbb{Z}/p\mathbb{Z}$ are Hensel zeros. We also require that Q has at least one zero in $\mathbb{Z}/p\mathbb{Z}$. The asymptotic zero number is defined (provided it exists) by the limit*

$$(3.1) \quad N_p(Q) := \lim_{n \rightarrow \infty} \frac{(p-1)\nu_p(t_n)}{n}.$$

Theorem 1.2 is restated as follows:

Theorem 3.2. *Let p be a Hensel prime for Q . Then $N_p(Q) = z_p$.*

Note. The examples will show pairs (Q, p) for which $N_p(Q) \notin \mathbb{N}$. An appropriate interpretation of this number is lacking in these cases.

In the examples described below we present the *normalized error*

$$(3.2) \quad \text{err}_p(n; Q) := z_p n - (p-1)\nu_p(t_n(Q))$$

and the *relative error*:

$$(3.3) \quad \text{relerr}_p(n; Q) := \text{err}_p(n; Q) - \text{err}_p(n-1; Q).$$

Certain regular structure of this function, as seen in Figure 3, will be analyzed in a future report.

Example 1. Let $Q(x) = x^5 + 2x^3 + 3$. Then $p = 5$ is a Hensel prime for Q . Indeed, the only zeros of Q in $\mathbb{Z}/5\mathbb{Z}$ are $a = 3$ and $a = 4$ and $Q'(a) \not\equiv 0 \pmod{5}$. Theorem 1.2 gives

$$(3.4) \quad \nu_5(t_n(Q)) \sim \frac{n}{2}.$$

Figure 1 shows the valuation $\nu_5(t_n(Q))$. Figure 2 and 3 depict patterns in the normal and relative error, respectively.

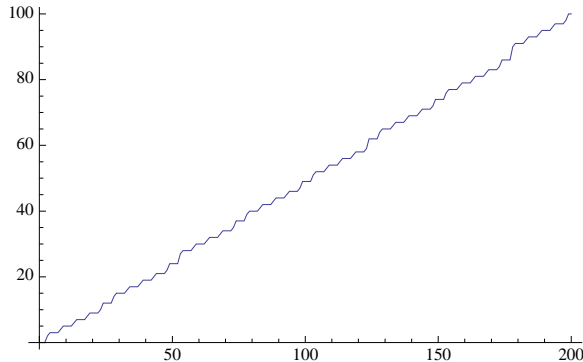


FIGURE 1. The valuation $\nu_5(t_n)$ for $Q(x) = x^5 + 2x^3 + 3$.

Example 2. A direct calculation shows that, among the first 20000 primes, $p = 3, 11$ and 29 are the only non-Hensel primes for $Q(x) = x^5 + 2x^3 + 3$. We now describe the asymptotic behavior of $\nu_p(t_n(Q))$ in each of these cases. The polynomial Q factors as

$$(3.5) \quad x^5 + 2x^3 + 3 = (x+1)H(x)$$

where

$$(3.6) \quad H(x) = x^4 - x^3 + 3x^2 - 3x + 3$$

and the valuation splits as

$$(3.7) \quad \nu_p(t_n(Q)) = \nu_p(t_n(x+1)) + \nu_p(t_n(H(x))).$$

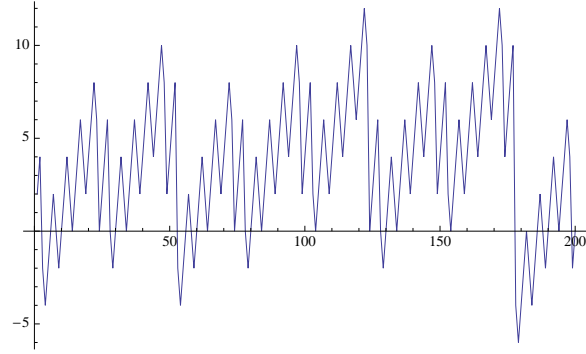


FIGURE 2. The normalized error when $p = 5$ and $Q(x) = x^5 + 2x^3 + 3$.

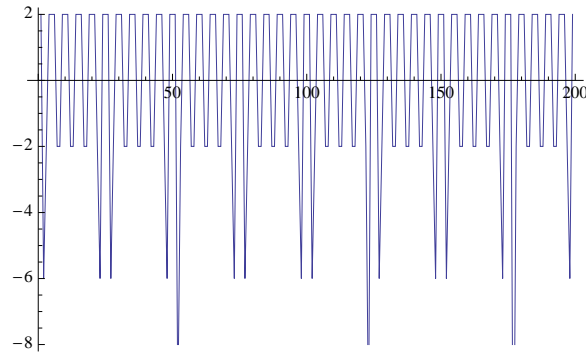


FIGURE 3. The relative error when $p = 5$ and $Q(x) = x^5 + 2x^3 + 3$.

Theorem 1.2 gives $\nu_p(t_n(x+1)) \sim n/(p-1)$, so it remains to evaluate $\nu_p(t_n(H))$.

The prime $p = 3$. In this case 0 and 1 are zeros of H in $\mathbb{Z}/3\mathbb{Z}$, and only 1 is a Hensel zero. Observe that

$$(3.8) \quad \nu_3(t_n(H)) = \sum_{j \equiv 0 \pmod{3}} \nu_3(H(j)) + \sum_{j \equiv 1 \pmod{3}} \nu_3(H(j)).$$

Since 1 is a Hensel zero, the argument in the proof of Theorem 1.2 implies that

$$(3.9) \quad \sum_{j \equiv 1 \pmod{3}} \nu_3(H(j)) \sim \frac{n}{2}.$$

To analyze the first sum in (3.8), note that

$$(3.10) \quad H(3k) = 81k^4 - 27k^3 + 27k^2 - 9k + 3.$$

Thus, $\nu_3(H(3k)) = 1$ for $k \in \mathbb{N}$. We obtain that

$$(3.11) \quad \sum_{j \equiv 0 \pmod{3}} \nu_3(H(j)) \sim \frac{n}{3},$$

and then $\nu_3(t_n(H)) \sim \frac{5n}{6}$. Therefore $\nu_3(t_n(Q)) \sim \frac{4n}{3}$ and $N_3(Q) = \frac{8}{3}$.

The prime $p = 11$. For this prime, although Theorem 1.2 does not apply to Q itself, it is applicable to both factors $x + 1$ and $H(x)$. And, we deduce

$$(3.12) \quad \nu_{11}(t_n(Q)) \sim \frac{n}{10} + \frac{2n}{10} = \frac{3n}{10}.$$

Therefore $N_{11}(Q) = 3$.

The prime $p = 29$. In order to find the asymptotic behavior of $\nu_{29}(t_n(H))$, observe that 14 is the only zero of H in $\mathbb{Z}/29\mathbb{Z}$ and

$$(3.13) \quad H(29k + 14) = 36221 + 303601k + 956217k^2 + 1341395k^3 + 707281k^4.$$

The valuations of the coefficients in $H(29k + 14)$ are 1, 2, 2, 3, and 4, respectively. Therefore $\nu_{29}(H(j)) = 1$ if $j \equiv 1 \pmod{29}$ and 0 otherwise. We conclude that

$$(3.14) \quad \nu_{29}(t_n(H)) \sim \frac{n}{29}.$$

Therefore $\nu_{29}(t_n(Q)) \sim \frac{57n}{812}$ and $N_{29}(Q) = \frac{57}{29}$.

Example 3. The polynomial

$$(3.15) \quad Q(x) = x^8 + x^5 + x^3 + 1 = (x^3 + 1)(x^5 + 1)$$

does not have a Hensel prime. This follows from

$$(3.16) \quad \gcd(Q(x), Q'(x)) = x + 1,$$

so that, for any prime p , we have that $p - 1$ is a zero of Q in $\mathbb{Z}/p\mathbb{Z}$ and $Q'(p - 1) = 0$. Naturally we have

$$(3.17) \quad \nu_p(t_n(Q)) = \nu_p(t_n(x^3 + 1)) + \nu_p(t_n(x^5 + 1)).$$

The asymptotic behavior of $\nu_p(t_n(Q))$ is discussed next.

Lemma 3.3. *Let p be an odd prime and $x \neq 1$. Then*

$$(3.18) \quad \nu_p(x^p - 1) = \begin{cases} 0 & \text{if } x \not\equiv 1 \pmod{p} \\ 1 + \nu_p(x - 1) & \text{if } x \equiv 1 \pmod{p}. \end{cases}$$

Proof. The first part is clear from the congruence $x^p \equiv x \pmod{p}$. To verify the second assertion, write $x = kp + 1$ and observe that

$$(3.19) \quad \nu_p(x^p - 1) = \nu_p \left(\sum_{r=1}^p \binom{p}{r} k^r p^r \right).$$

The p -adic valuation of each term in the sum is greater than $2 + \nu_p(k)$. When $r = 1$, it is exactly $2 + \nu_p(k)$. Then, putting $k = \frac{x-1}{p}$ verifies the assertion. \square

Corollary 3.4. *Let p be an odd prime and $x \in \mathbb{Z}$, $x \neq 1$. Define*

$$(3.20) \quad T_p(x) = x^{p-1} + x^{p-2} + \cdots + 1.$$

Then

$$(3.21) \quad \nu_p(T_p(x)) = \begin{cases} 0 & \text{if } x \not\equiv 1 \pmod{p} \\ 1 & \text{if } x \equiv 1 \pmod{p}. \end{cases}$$

Corollary 3.5. *Let p be a prime and $x \in \mathbb{Z}$, $x \neq -1$. Then*

$$(3.22) \quad \nu_p(x^p + 1) = \begin{cases} 0 & \text{if } x \not\equiv -1 \pmod{p} \\ 1 + \nu_p(x + 1) & \text{if } x \equiv -1 \pmod{p}. \end{cases}$$

Proof. Replace x by $-x$ in Lemma 3.3. □

Corollary 3.6. *Let p be a prime and $x \in \mathbb{Z}$, $x \neq -1$. Define*

$$(3.23) \quad S_p(x) = x^{p-1} - x^{p-2} + \cdots - x + 1.$$

Then

$$(3.24) \quad \nu_p(S_p(x)) = \begin{cases} 0 & \text{if } x \not\equiv -1 \pmod{p} \\ 1 & \text{if } x \equiv -1 \pmod{p}. \end{cases}$$

The number of roots of $x^q + 1 \equiv 0 \pmod{p}$, that is, $z_p(x^q + 1)$ stated in the Lemma below appears at the end of Section 8.1 of [3].

Lemma 3.7. *Let p and q be primes. The number of solutions of the congruence $x^p + 1 \equiv 0 \pmod{q}$ is $\gcd(p, q - 1)$.*

Corollary 3.8. *Let p be an odd prime. Then*

$$(3.25) \quad \nu_p(t_n(x^p \pm 1)) \sim \frac{(2p - 1)n}{p(p - 1)}.$$

If q is a prime, $q \neq p$, then

$$(3.26) \quad \nu_q(t_n(x^p \pm 1)) \sim \frac{\gcd(p, q - 1)n}{q - 1}.$$

Proof. Theorem 1.2 gives $\nu_p(t_n(x + 1)) \sim \frac{n}{p-1}$. The expression for $\nu_p(S_p(x))$ yields $\nu_p(S_p(x)) \sim n/p$. The asymptotic behavior of $\nu_q(t_n(x^p \pm 1))$ follow directly from Theorem 1.2. □

We now complete the analysis of

$$(3.27) \quad \nu_p(t_n(Q)) = \nu_p(t_n(x^3 + 1)) + \nu_p(t_n(x^5 + 1)).$$

If $p \neq 3$ is a prime, then

$$(3.28) \quad \nu_p(t_n(x^3 + 1)) \sim \frac{z_p(x^3 + 1)n}{p - 1}.$$

Similarly, for $p \neq 5$ prime, we have

$$(3.29) \quad \nu_p(t_n(x^5 + 1)) \sim \frac{z_p(x^5 + 1)n}{p-1}.$$

Thus, (3.25) and (3.29) yield

$$\nu_3(t_n(Q)) \sim \nu_3(t_n(x^3 + 1)) + \nu_3(t_n(x^5 + 1)) = \frac{5n}{6} + \frac{n}{2} = \frac{4n}{3}.$$

Similarly, $\nu_5(t_n(Q)) \sim 7n/10$.

Now let $p \neq 3, 5$ be a prime. Theorem 1.2 now applies directly to give

$$(3.30) \quad \nu_p(t_n(Q)) \sim \frac{[z_p(x^3 + 1) + z_p(x^5 + 1)]n}{p-1}.$$

Lemma 3.7 yields

$$(3.31) \quad \nu_p(t_n(Q)) \sim \frac{[\gcd(3, p-1) + \gcd(5, p-1)]n}{p-1}.$$

The asymptotic zero number is given by

$$(3.32) \quad N_p((x^3 + 1)(x^5 + 1)) = \begin{cases} \frac{8}{3} & \text{if } p = 3 \\ \frac{14}{5} & \text{if } p = 5 \\ \gcd(3, p-1) + \gcd(5, p-1) & \text{if } p \neq 3, 5. \end{cases}$$

Example 4. Let p be an arbitrary prime and define

$$(3.33) \quad A_p(x) = (px + 1)^2((p+1)x + 1).$$

A direct calculation shows that p is the only Hensel prime for A_p . Therefore

$$(3.34) \quad \nu_p(t_n(A_p)) \sim \frac{n}{p-1}.$$

To compute the asymptotics for a prime $q \neq p$, let $Q_1(x) = px + 1$ and $Q_2(x) = (p+1)x + 1$, and observe that

$$(3.35) \quad \nu_q(t_n(A_p)) = 2\nu_q(t_n(Q_1)) + \nu_q(t_n(Q_2)).$$

Theorem 1.2 applies to both Q_1 and Q_2 . The case for Q_1 is immediate since $px + 1 \equiv 0 \pmod{q}$ has a unique solution. To evaluate $\nu_q(t_n(Q_2))$ observe that the number of solutions of $(p+1)x + 1 \equiv 0 \pmod{q}$ is 0 or 1, according to whether q divides $p+1$ or not. Thus

$$(3.36) \quad \nu_q(t_n(A_p)) \sim \frac{(2 + \omega_{p,q})n}{q-1}$$

where

$$\omega_{p,q} = \begin{cases} 1 & \text{if } q \text{ divides } p+1, \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that

$$(3.37) \quad N_q(A_p) = \begin{cases} 1 & \text{if } p = q, \\ 2 + \omega_{p,q} & \text{if } p \neq q. \end{cases}$$

4. FUTURE DIRECTIONS

In this section we outline certain generalizations of the main result of the paper.

A natural extension of Theorem 1.2 deals with the situation in which there is an element $b \in \mathbb{Z}/p\mathbb{Z}$ such that

$$(4.1) \quad Q(b) \equiv Q'(b) \equiv \dots \equiv Q^{(k-1)}(b) \equiv 0 \pmod{p}.$$

The question of how the multiplicities of the roots enter in the asymptotic behavior of $\nu_p(t_n(Q))$ appears to be a salient quest, and this will be addressed elsewhere.

Another interesting continuation of the ideas presented in this paper would be the study of p -adic valuation of sequences satisfying second order recurrences

$$(4.2) \quad t_n = Q_1(n)t_{n-1} + Q_2(n)t_{n-2},$$

with polynomials Q_1 and Q_2 . This problem includes, classically, the case of Fibonacci and Stirling numbers.

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