THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 19: EVALUATION USING BESSEL FUNCTIONS. PRELIMINARY VERSION: LAST UPDATE OCTOBER 3, 2006.

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Abstract. The table of Gradshteyn and Rhyzik contains many integrals that can be evaluated using the Bessel function. Some examples are discussed.

1. Introduction

The Bessel function J_ν is defined by

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k}. \tag{1.1}$$

and similarly

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k}. \tag{1.2}$$

In particular,

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}\,k!^2} z^{2k} \tag{1.3}$$

and

$$I_0(z) = \sum_{k=0}^{\infty} \frac{1}{2^{2k}\,k!^2} z^{2k}. \tag{1.4}$$

2. An exponential representation

The evaluation of 3.339 in [1]:

$$\int_0^\pi e^{z\,\cos x}\,dx = \pi\,I_0(z) \tag{2.1}$$

• 3.339 can be done directly by expanding the exponential

$$I = \sum_{j=0}^{\infty} \frac{z^j}{j!} \int_0^\pi \cos^j z\,dz. \tag{2.2}$$

The integral vanishes for j odd and for $j=2k$, using Wallis's formula

$$\int_0^\pi \cos^{2k} x\,dx = \frac{\pi}{2^{2k}} \frac{(2k)!}{k!^2} \tag{2.3}$$

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yields the result.

3. An integral representation

The integral representations of the Bessel functions are stated next.

Theorem 3.1. *The Bessel function* $J_\nu(z)$ *has the representation*

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} \cos zt\, dt. \tag{3.1}$$

Similarly,

$$I_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} e^{zt}\, dt. \tag{3.2}$$

Proof. Start ☐

In particular, for $\nu = 0$, we have

$$J_0(z) = \frac{1}{\pi} \int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} \cos zt\, dt, \tag{3.3}$$

and

$$I_0(z) = \frac{1}{\pi} \int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} e^{zt}\, dt. \tag{3.4}$$

The special case $\nu = 0$ and $z = 2$ gives 3.364.2:

$$\int_{-1}^{1} \frac{e^{2x}\, dx}{\sqrt{1-x^2}} = \pi I_0(2). \tag{3.5}$$

The identity • 3.364.2

$$\int_{0}^{2} \frac{e^{-px}\, dx}{\sqrt{x(2-x)}} = e^{-p} \int_{-1}^{1} \frac{e^{-pt}\, dx}{\sqrt{1-t^2}}, \tag{3.6}$$

comes from the change of variables $t = x - 1$. Changing t to $-t$ gives the evaluation of 3.364.1: • 3.364.1

$$\int_{0}^{2} \frac{e^{-px}\, dx}{\sqrt{x(2-x)}} = \pi e^{-p} I_0(p). \tag{3.7}$$

4. The Bessel K-function

The Bessel function of imaginary argument $K_\nu(z)$ defined by

$$K_\nu(z) = ?? \tag{4.1}$$

admits the integral representation

$$K_\nu(z) = \frac{z^\nu}{2^{\nu+1}} \int_{0}^{\infty} t^{-\nu-1} e^{-t-z^2/4t}\, dt. \tag{4.2}$$

This formula appears in 8.432.6. The change of variables $s = 1/t$ yields

$$K_\nu(z) = \frac{z^\nu}{2^{\nu+1}} \int_{0}^{\infty} s^{\nu-1} e^{-1/s-z^2s/4}\, ds. \tag{4.3}$$

The change of variables $s = aw$ produces

$$K_\nu(z) = \frac{z^\nu a^\nu}{2^{\nu+1}} \int_{0}^{\infty} w^{\nu-1} e^{-1/as-bs}\, ds, \tag{4.4}$$

with $z = 2\sqrt{b}/\sqrt{a}$. The special case $\nu = 1$, with $c = 4/a$ yields

$$\int_0^\infty e^{-c/4s-bs}\,ds = \sqrt{\frac{c}{b}}K_1(\sqrt{bc}). \tag{4.5}$$

- 3.324.1 This appears as 3.324.1 in [1].

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References

[1] I.S. Gradshteyn and I.M. Rhyzik. *Table of Integrals, Series, and Products.* Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.

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