THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 13: EVALUATION USING THE ERROR FUNCTION. PRELIMINARY VERSION: LAST UPDATE OCTOBER 4, 2006.

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Abstract. The table of Gradshteyn and Rhyzik contains many integrals that can be evaluated using the error function. Some examples are discussed.

1. Introduction

The *error* function is defined by

$$\text{erf}(u) := \frac{2}{\sqrt{\pi}} \int_0^u e^{-x^2}\, dx. \tag{1.1}$$

The value $2/\sqrt{\pi}$ is a normalization factor that has the effect of giving $\text{erf}(\infty) = 1$ in view of the *normal integral*

$$\int_0^\infty e^{-x^2}\, dx = \frac{2}{\sqrt{\pi}}. \tag{1.2}$$

The reader will find in [1] different proofs of this evaluation.

The table [2] contains many integrals involving the error function. The first identity for it is 3.321.1:

$$\text{erf}(u) = \frac{2}{\sqrt{\pi}} e^{-u^2} \sum_{k=0}^\infty \frac{2^k}{(2k+1)!!} u^k. \tag{1.3}$$

• 3.321.1 To check this identity we need to prove

$$\left(\sum_{k=0}^\infty \frac{(-1)^k}{k!} u^{2k} \right) \times \left(\sum_{j=0}^\infty \frac{2^j}{(2j+1)!!} u^{2j+1} \right) = \sum_{r=0}^\infty \frac{(-1)^r}{r!(2r+1)}.$$

Multiplying the two series on the left, we conclude that the result follows from the finite sums identity

$$\sum_{k=0}^r \frac{(-1)^k 2^{r-k}}{k!\,(2r-2k+1)!!} = \frac{(-1)^r}{r!\,(2r+1)}. \tag{1.4}$$

Write this identity as

$$\sum_{k=0}^r (-4)^k \binom{r}{k} \binom{2k}{k}^{-1} \frac{2r+1}{2k+1} = 1. \tag{1.5}$$

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We prove this using the WZ-technology [3]: define

$$A(r,k) = (-4)^k \binom{r}{k} \binom{2k}{k}^{-1} \frac{2r+1}{2k+1} \tag{1.6}$$

and define (this comes from the WZ-method) the function

$$B(r,k) = (-1)^{k+1} \binom{r}{k-1} \binom{2k}{k}^{-1}. \tag{1.7}$$

Then a direct computation verifies

$$A(r+1,k) - A(r,k) = B(r,k+1) - B(r,k). \tag{1.8}$$

Now sum from $k = -\infty$ to $k = +\infty$ to prove that

$$a_r := \sum_{k=0}^{r} A(r,k) \tag{1.9}$$

is independent of r. The value $a_0 = 1$ completes the proof.

Thanks. The author thanks T. Amdeberham for supplying this proof.

2. Elementary scaling

The table [2] contains many integrals involving the error function. For instance, the change of variables $x = qt$ yields

$$\text{erf}(u) = \frac{2q}{\sqrt{\pi}} \int_0^{u/q} e^{-q^2t^2} \, dt. \tag{2.1}$$

Replacing u by qu yields 3.321.2: • 3.321.2

$$\int_0^u e^{-q^2x^2} \, dx = \frac{\sqrt{\pi}}{2q} \text{erf}(qu). \tag{2.2}$$

Now let $u \to \infty$ and use $\text{erf}(\infty) = 1$ to produce 3.321.3: • 3.321.3

$$\int_0^\infty e^{-q^2x^2} \, dx = \frac{\sqrt{\pi}}{2q}. \tag{2.3}$$

Simple scaling produces other integrals. Starting with

$$\int_a^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \left(1 - \text{erf}(a)\right), \tag{2.4}$$

the change of variables $t = qx$ yields

$$\int_a^\infty e^{-q^2x^2} \, dx = \frac{\sqrt{\pi}}{2q} \left(1 - \text{erf}(qa)\right), \tag{2.5}$$

The integral

$$\int_a^\infty e^{-q^2x^2 - px} \, dx = \frac{\sqrt{\pi}}{2q} e^{p^2/4q^2} \left(1 - \ \text{erf}\left(\frac{p + 2aq^2}{2q}\right)\right), \tag{2.6}$$

is computed by completing the square. The choice $q = 1/2\sqrt{\beta}$ and $p = \gamma$ appear as 3.322.1 in [2]: • 3.322.1

$$\int_u^\infty \exp\left(-\frac{x^2}{4\beta} - \gamma x\right) \, dx = \sqrt{\pi\beta} e^{\beta\gamma^2} \left(1 - \text{erf}\left(\frac{u}{2\sqrt{\beta}} + \sqrt{\beta}\gamma\right)\right). \tag{2.7}$$

• 3.322.2 We prefer the notation in (2.6). Letting $a \to 0$ produces 3.322.2:

$$\int_0^\infty e^{-q^2x^2-px}\,dx = \frac{\sqrt{\pi}}{2q}e^{p^2/4q^2}\left(1 - \text{ erf}\left(\frac{p}{2q}\right)\right). \tag{2.8}$$

Now let $q = 1$ and $a = 1$ in (2.6) to produce

$$\int_1^\infty e^{-x^2-px}\,dx = \frac{\sqrt{\pi}}{2}e^{p^2/4}\left(1 - \text{ erf}\left(\frac{p+2}{2}\right)\right), \tag{2.9}$$

• 3.323.1 This appears as 3.323.1 (unfortunately with p instead of q).

• 3.323.2 The evaluation 3.323.2:

$$\int_{-\infty}^\infty \exp\left(-p^2x^2 \pm qx\right)\,dx = \frac{\sqrt{\pi}}{p}\exp\left(\frac{q^2}{4p^2}\right) \tag{2.10}$$

follows directly from completing the square, as in

$$-p^2x^2 \pm qx = -p^2\left(x \mp q/2p^2\right)^2 + \frac{q^2}{4p^2} \tag{2.11}$$

3. An integral of Laplace

• 3.325 Laplace produced the evaluation of 3.325:

$$\int_0^\infty \exp\left(-ax^2 - bx^{-2}\right)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}. \tag{3.1}$$

To evaluate this, we complete square in the exponent and write

$$\int_0^\infty \exp\left(-ax^2 - bx^{-2}\right)\,dx = e^{-2\sqrt{ab}}\int_0^\infty e^{-(\sqrt{a}x-\sqrt{b}/x)^2}\,dx. \tag{3.2}$$

Denote this last integral by J, that is,

$$J := \int_0^\infty e^{-(\sqrt{a}x-\sqrt{b}/x)^2}\,dx. \tag{3.3}$$

The change of variables $t = \sqrt{b}/\sqrt{a}x$ produces

$$J := \frac{\sqrt{b}}{\sqrt{a}}\int_0^\infty e^{-(\sqrt{a}t-\sqrt{b}/t)^2}\,\frac{dt}{t^2}. \tag{3.4}$$

The average of these two forms for J produces

$$J = \frac{1}{2\sqrt{a}}\int_0^\infty e^{-(\sqrt{a}x-\sqrt{b}/x)^2}\left(\sqrt{a} + \sqrt{b}/x^2\right)\,dx. \tag{3.5}$$

The change of variables $u = \sqrt{a}x - \sqrt{b}/x$ now yields

$$J = \frac{1}{2\sqrt{a}}\int_{-\infty}^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2\sqrt{a}}, \tag{3.6}$$

and the evaluation is complete.

The example 3.472.1:

$$\int_0^\infty \left(\exp\left(-a/x^2\right) - 1\right)\,e^{-\mu x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}\left(e^{-2\sqrt{a\mu}} - 1\right) \tag{3.7}$$

• 3.472.1 can be evaluated directly from 3.325. Indeed,

$$\int_0^\infty \left(\exp\left(-a/x^2\right)-1\right) e^{-\mu x^2}\,dx = \int_0^\infty \exp\left(-a/x^2-\mu x^2\right)\,dx - \int_0^\infty e^{-\mu x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}\left(e^{-2\sqrt{a\mu}}-1\right),$$

as required.

The example 3.471.15:

$$\int_0^\infty x^{-1/2}e^{-ax-b/x}\,dx = \sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}} \tag{3.8}$$

can be reduced, via $t = x^{1/2}$, to • 3.471.15

$$I = 2\int_0^\infty e^{-at^2-b/t^2}\,dt. \tag{3.9}$$

The value of this integral is given in (3.1).

Differentiating with respect to the parameter p shows that the integral

$$I_n(p) = \int_0^\infty x^{n-1/2}e^{-px-q/x}\,dx \tag{3.10}$$

satisfies

$$\frac{\partial I_n}{\partial p} = -I_{n+1}(p). \tag{3.11}$$

Using this it is an easy induction exercise to verify the evaluation

$$\int_0^\infty x^{n-1/2}e^{-px-q/x}\,dx = (-1)^n\sqrt{\pi}\left(\frac{\partial}{\partial p}\right)^n\left[p^{-1/2}e^{-2\sqrt{pq}}\right]. \tag{3.12}$$

This is entry 3.471.16 of [2]. • 3.471.16

4. Some elementary changes of variables

The change of variables $x = \sqrt{tq}$ in the integral

$$\operatorname{erf}(u) := \frac{2}{\sqrt{\pi}}\int_0^u e^{-x^2}\,dx \tag{4.1}$$

produces

$$\int_0^{u^2/q} \frac{e^{-qt}}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{q}}\operatorname{erf}(u). \tag{4.2}$$

Now let $v = u^2/q$ to produce • 3.361.1

$$\int_0^v \frac{e^{-qt}}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{q}}\operatorname{erf}(\sqrt{qv}). \tag{4.3}$$

This appears as 3.361.1 in [2]. Letting $v \to \infty$ and using $\operatorname{erf}(+\infty) = 1$, we obtain 3.361.2: • 3.361.2

$$\int_0^\infty \frac{e^{-qt}}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{q}}. \tag{4.4}$$

The change of variables $x = t - a$ produces

$$\int_a^\infty \frac{e^{-qt}}{\sqrt{t-a}}\,dt = e^{-aq}\sqrt{\frac{\pi}{q}}. \tag{4.5}$$

The special case $a = -1$ appears as 3.361.3:

$$\int_{-1}^{\infty} \frac{e^{-qt}}{\sqrt{t+1}}\, dt = e^{q}\sqrt{\frac{\pi}{q}}, \tag{4.6}$$

• 3.361.3 and $a = 1$ appears as 3.362.1:

$$\int_{1}^{\infty} \frac{e^{-qt}}{\sqrt{t-1}}\, dt = e^{-q}\sqrt{\frac{\pi}{q}}. \tag{4.7}$$

• 3.362.1 The integral in 3.362.2:

$$\int_{0}^{\infty} \frac{e^{-qt}}{\sqrt{t+b}}\, dt = \sqrt{\frac{\pi}{q}}e^{bq}\left(1 - \operatorname{erf}(qb)\right) \tag{4.8}$$

• 3.362.2 can also be established by elementary means. Indeed, the change of variables $t = s^2 - b$ yields

$$I = 2e^{qb}\int_{\sqrt{b}}^{\infty} e^{-qs^2}\, ds, \tag{4.9}$$

and scaling by $s = \sqrt{q}y$ yields

$$I = \frac{2e^{qb}}{\sqrt{q}}\int_{\sqrt{qb}}^{\infty} e^{-y^2}\, dy \tag{4.10}$$

that can be written as

$$I = \frac{2e^{qb}}{\sqrt{q}}\left(\sqrt{\frac{\pi}{2}} - \int_{0}^{\sqrt{qb}} e^{-y^2}\, dy\right), \tag{4.11}$$

and now just write this in terms of the error function to get the stated result.

The evaluation of 3.461.5:

$$\int_{u}^{\infty} e^{-qx^2}\frac{dx}{x^2} = \frac{1}{u}e^{-qu^2} - \sqrt{\pi q}\left(1 - \operatorname{erf}(u\sqrt{q})\right), \tag{4.12}$$

• 3.461.5 is obtained by integration by parts. Indeed

$$\int_{u}^{\infty} e^{-qx^2}\frac{dx}{x^2} = \frac{1}{u}e^{-qu^2} - 2q\int_{u}^{\infty} e^{-qx^2}\, dx, \tag{4.13}$$

and this last integral can be reduced to the error function using

$$\int_{u}^{\infty} e^{-qx^2}\, dx = \int_{0}^{\infty} e^{-qx^2}\, dx - \int_{0}^{u} e^{-qx^2}\, dx. \tag{4.14}$$

The evaluation of 3.466.2:

$$\int_{0}^{\infty} \frac{x^2 e^{-a^2x^2}}{x^2+b^2}\, dx = \frac{\sqrt{\pi}}{2a} - \frac{\pi b}{2}e^{a^2b^2}\left(1 - \operatorname{erf}(ab)\right) \tag{4.15}$$

• 3.466.6

is obtained by writing

$$\frac{\partial}{\partial a}\left(Ie^{-a^2b^2}\right) = -2\mu e^{-a^2b^2}\int_{0}^{\infty} x^2 e^{-a^2x^2}\, dx$$

and the integral can be evaluated via the change of variables $t = ax$, to get

$$\frac{\partial}{\partial a}\left(Ie^{-a^2b^2}\right) = -\frac{\sqrt{\pi}}{2}\frac{e^{-a^2b^2}}{b^2}.$$

Now integrate from a to ∞ and use 3.461.5 to obtain

$$-Ie^{-a^2b^2} = -\frac{\sqrt{\pi}}{2}\left(\frac{e^{-a^2b^2}}{a} - b\sqrt{\pi}\,(1-\text{erf}(ab))\right). \tag{4.16}$$

Now simplify to produce the result.

Similarly, 3.462.5:

$$\int_0^\infty xe^{-\mu x^2-2\nu x}\,dx = \frac{1}{2\mu} - \frac{\nu}{2\mu}\sqrt{\frac{\pi}{\mu}}(1-\text{erf}(\nu/\sqrt{\mu})) \tag{4.17}$$

• 3.462.5

can be evaluated in elementary terms. The change of variables $t = \sqrt{\mu}x$ followed by $y = t + c$ with $c = \nu/\sqrt{\mu}$ yields

$$I = \frac{e^{c^2}}{\mu}J \tag{4.18}$$

where

$$J = \int_c^\infty (y-c)e^{-y^2}\,dy. \tag{4.19}$$

The first integrand is a perfect derivative and the second one can be reduced to the error function to complete the evaluation.

The integral in 3.462.6:

$$\int_{-\infty}^\infty xe^{-px^2-2qx}\,dx = \frac{q}{p}\sqrt{\frac{\pi}{p}}\exp(q^2/p) \tag{4.20}$$

is evaluated by completing the square in the exponent. It produces • 3.462.6

$$I = e^{q^2/p}\int_{-\infty}^\infty xe^{-p(x-q/p)^2}\,dx \tag{4.21}$$

and shifting the integrand by $t = x - p/q$ yields

$$I = e^{q^2/p}\int_{-\infty}^\infty (t+p/q)e^{-pt^2}\,dt \tag{4.22}$$

The first integral is elementary and the second one can be reduced to the error function to produce the result.

Similar arguments give 3.462.7: • 3.462.7

$$\int_0^\infty x^2e^{-\mu x^2-2\nu x}\,dx = -\frac{\nu}{2\mu^2} + \sqrt{\frac{\pi}{\mu^5}}\frac{2\nu^2+\mu}{4}e^{\nu^2/\mu}(1-\text{erf}(\nu/\sqrt{\mu})), \tag{4.23}$$

and 3.462.8:

• 3.462.8

$$\int_{-\infty}^\infty x^2e^{-\mu x^2+2\nu x}\,dx = \frac{1}{2\mu}\sqrt{\frac{\pi}{\mu}}(1+2\nu^2/\mu)e^{\nu^2/\mu}. \tag{4.24}$$

5. Some more challenging integrals

The example 3.363.1:

$$\int_u^\infty \frac{\sqrt{x-u}}{x} e^{-qx}\, dx = \sqrt{\frac{\pi}{q}} e^{-qu} - \pi\sqrt{u}(1-\operatorname{erf}(\sqrt{qu})) \tag{5.1}$$

• 3.363.1 is elementary, but our evaluation is more complicated than those in the previous section.

We first let $x = u + t^2$ to produce $I = 2e^{-qu}J$, where

$$J = \int_0^\infty \frac{t^2}{t^2+u} e^{-qt^2}\, dt. \tag{5.2}$$

The next step is to write

$$J = \int_0^\infty e^{-qt^2}\, dt - u\int_0^\infty \frac{e^{-qt^2}}{u+t^2} dt. \tag{5.3}$$

The first integral evaluates as $\sqrt{\pi}/2\sqrt{q}$ and we let

$$K = \int_0^\infty \frac{e^{-r^2}}{r^2+qu}\, dr, \tag{5.4}$$

so that

$$I = \sqrt{\frac{\pi}{q}} e^{-qu} - 2u\sqrt{q}e^{-qu}K. \tag{5.5}$$

The change of variables $s = r^2 = qu$ produces, with $\omega = qu$,

$$K = \frac{1}{2}e^{qu}\int_\omega^\infty \frac{e^{-s}}{s\,\sqrt{s-\omega}}. \tag{5.6}$$

The scaling $s = \omega y$ reduces the question to the evaluation of

$$T = \int_1^\infty \frac{e^{-\omega y}}{y\,\sqrt{y-1}}\, dy. \tag{5.7}$$

Observe that

$$\frac{\partial T}{\partial \omega} = -\int_1^\infty \frac{e^{-\omega y}}{\sqrt{y-1}}\, dy = -\sqrt{\frac{\pi}{\omega}} e^{-\omega}, \tag{5.8}$$

where we have used 3.362.1. Integrate back to produce

$$\begin{aligned} T &= \sqrt{\pi}\int_\omega^\infty \frac{e^{-r}}{\sqrt{r}} dr \\ &= \sqrt{\pi}\left(\int_0^\infty \frac{e^{-r}}{\sqrt{r}} dr - \int_0^\omega \frac{e^{-r}}{\sqrt{r}}\, dr\right) \\ &= \pi(1-\operatorname{erf}(\sqrt{\omega})). \end{aligned}$$

This gives the stated result.

Using the identity

$$\frac{1}{x\,\sqrt{x-u}} = \frac{1}{\sqrt{x-u}} - \frac{u}{x\sqrt{x-u}} \tag{5.9}$$

and the results of 3.362.2 and 3.363.1 we obtain 3.363.2:

$$\int_u^\infty \frac{e^{-qx}\, dx}{x\sqrt{x-u}} = \frac{\pi}{\sqrt{u}}\left(1-\operatorname{erf}(\sqrt{qu})\right). \tag{5.10}$$

• 3.363.2

6. Differentiation with respect to a parameter

The evaluation of 3.466.1:

$$\int_0^\infty \frac{e^{-a^2x^2}\,dx}{x^2+b^2} = \frac{\pi}{2b}(1-\text{erf}(ab))e^{a^2b^2}, \tag{6.1}$$

• 3.466.1

can be simplified by the scaling $x = bt$. This yields the equivalent form

$$\int_0^\infty \frac{e^{-c^2t^2}\,dt}{1+t^2} = \frac{\pi}{2}(1-\text{erf}(c))e^{c^2}, \tag{6.2}$$

with $c = ab$. Introduce the function

$$f(c) = \int_0^\infty \frac{e^{-c^2(1+t^2)}}{1+t^2}\,dt \tag{6.3}$$

and the identity is equivalent to proving

$$f(c) = \frac{\pi}{2}(1-\text{erf}(c)). \tag{6.4}$$

Differentiation with respect to c we get

$$f'(c) = -2ce^{-c^2}\int_0^\infty e^{-(ct)^2}\,dt = -\sqrt{\pi}e^{-c^2}. \tag{6.5}$$

Using the value $f(0) = \frac{\pi}{2}$ we get

$$f(c) = \frac{\pi}{2}(1-\text{erf}(c)) \tag{6.6}$$

as required.

The evaluation of 3.464:

$$\int_0^\infty \left(e^{-\mu x^2} - e^{-\nu x^2}\right)\frac{dx}{x^2} = \sqrt{\pi}\left(\sqrt{\nu}-\sqrt{\mu}\right), \tag{6.7}$$

is obtained by introducing

$$f(\mu) = \int_0^\infty \left(e^{-\mu x^2} - e^{-\nu x^2}\right)\frac{dx}{x^2} \tag{6.8}$$

and differentiating with respect to the parameter μ we obtain

$$f'(\mu) = -\int_0^\infty e^{-\mu x^2}\,dx = -\frac{\sqrt{\pi}}{2\sqrt{\mu}}. \tag{6.9}$$

Integrating back and using $f(\nu) = 0$ we obtain the result.

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References

[1] G. Boros and V. Moll. *Irresistible Integrals.* Cambridge University Press, New York, 1st edition, 2004.

[2] I.S. Gradshteyn and I.M. Rhyzik. *Table of Integrals, Series, and Products.* Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.

[3] M. Petkovsek, H. Wilf, and D. Zeilberger. *A=B.* A. K. Peters, Ltd., 1st edition, 1996.

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