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The integrals in Gradshteyn and Ryzhik. Part 13: Trigonometric forms of the beta function

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Abstract. The table of Gradshteyn and Rhyzik contains some trigonometric integrals that can be expressed in terms of the beta function. We describe the evaluation of some of them.

1. Introduction

The table of integrals [**2**] contains a large variety of definite integrals in trigonometric form that can be evaluated in terms of the *beta function* defined by

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx. \tag{1.1}$$

The convergence of the integral requires $a, b > 0$.

The change of variables $x = \sin^2 t$ yields the basic representation

$$B(a,b) = 2\int_0^{\pi/2} \sin^{2a-1} t\, \cos^{2b-1} t\, dt, \tag{1.2}$$

that, after replacing $(2a, 2b)$ by (a, b), is written as

$$\int_0^{\pi/2} \sin^{a-1} t\, \cos^{b-1} t\, dt = \frac{1}{2} B\left(\frac{a}{2}, \frac{b}{2}\right). \tag{1.3}$$

This appears as **3.621.5** in [**2**].

2. Special cases

In this section we present several special cases of formula (1.3) that appear in [**2**].

Example 2.1. The choice $b = 1$ in (1.3) gives

$$\int_0^{\pi/2} \sin^{a-1} t\, dt = \frac{1}{2} B\left(\frac{a}{2}, \frac{1}{2}\right). \tag{2.1}$$

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Legendre's duplication formula

$$\Gamma(2a) = \frac{2^{2a-1}}{\sqrt{\pi}} \Gamma(a)\Gamma(a+\tfrac{1}{2}) \tag{2.2}$$

can be used to write (2.1) as

$$\int_0^{\pi/2} \sin^{a-1} t \, dt = 2^{a-2} B\left(\frac{a}{2}, \frac{a}{2}\right) = \frac{2^{a-2}\,\Gamma^2(a/2)}{\Gamma(a)}. \tag{2.3}$$

This is **3.621.1** in [**2**]. The dual evaluation

$$\int_0^{\pi/2} \cos^{a-1} t \, dt = 2^{a-2} B\left(\frac{a}{2}, \frac{a}{2}\right) = \frac{2^{a-2}\,\Gamma^2(a/2)}{\Gamma(a)}, \tag{2.4}$$

comes from the change of variables $t \mapsto \frac{\pi}{2} - t$. The reader will find a proof of (2.2) in [**1**].

Example 2.2. The special case $a = \frac{1}{2}$ in (2.3) gives **3.621.7**:

$$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}}. \tag{2.5}$$

Example 2.3. The special case $a = \frac{3}{2}$ in (2.3) gives **3.621.6**:

$$\int_0^{\pi/2} \sqrt{\sin x} \, dx = \sqrt{\frac{2}{\pi}} \Gamma^2\left(\tfrac{1}{4}\right). \tag{2.6}$$

Example 2.4. The special case $a = \frac{5}{2}$ in (2.3) gives **3.621.2**:

$$\int_0^{\pi/2} \sin^{3/2} x \, dx = \frac{1}{6\sqrt{2\pi}} \Gamma^2\left(\tfrac{1}{4}\right). \tag{2.7}$$

Example 2.5. The special case $a = 2m+1$ in (2.3) gives

$$\int_0^{\pi/2} \sin^{2m} x \, dx = 2^{2m-1} B\left(m+\tfrac{1}{2}, m+\tfrac{1}{2}\right), \tag{2.8}$$

and using the identity

$$\Gamma\left(m+\tfrac{1}{2}\right) = \frac{\pi}{2^{2m}} \frac{(2m)!}{m!} \tag{2.9}$$

it yields

$$\int_0^{\pi/2} \sin^{2m} x \, dx = \frac{\binom{2m}{m}\pi}{2^{2m+1}}. \tag{2.10}$$

This appears as **3.621.3**. Similarly, $a = 2m+2$ in (2.3) gives

$$\int_0^{\pi/2} \sin^{2m+1} x \, dx = 2^{2m} B(m+1, m+1), \tag{2.11}$$

that can be written as

$$\int_0^{\pi/2} \sin^{2m+1} x \, dx = \frac{2^{2m}}{(2m+1)} \binom{2m}{m}^{-1}. \tag{2.12}$$

This is **3.621.4**.

Example 2.6. The integral **3.622.1** is

$$\begin{aligned}\int_0^{\pi/2} \tan^{\pm a} x\, dx &= \int_0^{\pi/2} \sin^{\pm a} x \, \cos^{\mp a} x\, dx \\ &= \tfrac{1}{2} B\left(\tfrac{1\pm a}{2}, 1 - \tfrac{1\pm a}{2}\right) \\ &= \tfrac{1}{2}\Gamma\left(\tfrac{1\pm a}{2}\right)\Gamma\left(1 - \tfrac{1\pm a}{2}\right)\end{aligned}$$

and this reduces to

$$\int_0^{\pi/2} \tan^{\pm a} x\, dx = \frac{\pi}{2\,\cos(\pi a/2)},$$

as it appears in the table.

Example 2.7. The identity

$$\tan^{a-1} x \, \cos^{2b-2} x = \sin^{a-1} x \, \cos^{2b-a-1} x \tag{2.13}$$

shows that

$$\int_0^{\pi/2} \tan^{a-1} x \, \cos^{2b-2} x\, dx = \int_0^{\pi/2} \sin^{a-1} x \, \cos^{2b-a-1} x\, dx = \frac{1}{2} B\left(\frac{a}{2}, b - \frac{a}{2}\right). \tag{2.14}$$

This appears as **3.623.1**.

Example 2.8. The formula **3.624.2** states that

$$\int_0^{\pi/2} \frac{\sin^{a-1/2} x}{\cos^{2a-1} x}\, dx = \frac{\Gamma\left(\frac{a}{2}+\frac{1}{4}\right)\Gamma(1-a)}{2\Gamma\left(\frac{5}{4}-\frac{a}{2}\right)}. \tag{2.15}$$

This comes directly from (1.3).

Example 2.9. The identity **3.627**:

$$\int_0^{\pi/2} \frac{\tan^a x}{\cos^a x}\, dx = \int_0^{\pi/2} \frac{\cot^a x}{\sin^a x}\, dx = \frac{\Gamma(a)\Gamma(\frac{1}{2}-a)}{2^a\sqrt{\pi}} \sin\left(\frac{\pi a}{2}\right), \tag{2.16}$$

can be verified by writing the first integral as

$$I = \int_0^{\pi/2} \sin^a x \, \cos^{1-2a} x\, dx = \frac{1}{2} B\left(\frac{a+1}{2}, \frac{1-2a}{2}\right). \tag{2.17}$$

The beta function is

$$\frac{1}{2} B\left(\frac{a+1}{2}, \frac{1-2a}{2}\right) = \frac{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-a\right)}{2\Gamma\left(1-\frac{a}{2}\right)}. \tag{2.18}$$

Using $\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin \pi t}$ we can reduce (2.18) to the expression in (2.16).

Example 2.10. The evaluation of **3.628**

$$\int_0^{\pi/2} \sec^{2p} x \, \sin^{2p-1} x\, dx = \frac{\Gamma(p)\Gamma(\frac{1}{2}-p)}{2\sqrt{\pi}}, \tag{2.19}$$

is direct, once we write the integral as

$$\int_0^{\pi/2} \cos^{-2p} x \sin^{2p-1} x\, dx = \frac{1}{2} B\left(\tfrac{1}{2}-p, p\right). \tag{2.20}$$

3. A family of trigonometric integrals

In this section we present the evaluation of a family of trigonometrical integrals in [**2**]. Many special cases appear in the table.

Proposition 3.1. Let a, b, $c \in \mathbb{R}$ with the condition

$$a + b + 2c + 2 = 0. \tag{3.1}$$

Then

$$\int_0^{\pi/4} \sin^a x \, \cos^b x \, \cos^c (2x) \, dx = \frac{1}{2} B\left(\frac{a+1}{2}, c+1 \right). \tag{3.2}$$

Proof. Let $t = \tan x$ to obtain

$$\int_0^{\pi/4} \sin^a x \, \cos^b x \, \cos^c(2x) \, dx = \int_0^1 t^a (1-t^2)^c (1+t^2)^{-(a+b+2c+2)/2} \, dt \tag{3.3}$$

and (3.1) yields

$$\int_0^{\pi/4} \sin^a x \, \cos^b x \, \cos^c(2x) \, dx = \int_0^1 t^a (1-t^2)^c \, dt. \tag{3.4}$$

The change of variables $s = t^2$ produces

$$\int_0^{\pi/4} \sin^a x \, \cos^b x \, \cos^c(2x) \, dx = \frac{1}{2} \int_0^1 s^{(a-1)/2} (1-s)^c \, ds, \tag{3.5}$$

and this last integral has the given beta value. □

Example 3.2. The formula (3.2), with $a = 2n$, $b = -2p - 2n - 2$ and $c = p$ appears as **3.625.2** in [**2**]:

$$\int_0^{\pi/4} \frac{\sin^{2n} x \, \cos^p(2x)}{\cos^{2p+2n+2} x} \, dx = \tfrac{1}{2} B\left(n + \tfrac{1}{2}, p+1\right). \tag{3.6}$$

Example 3.3. The formula **3.624.3**

$$\int_0^{\pi/4} \frac{\cos^{n-1/2}(2x)}{\cos^{2n+1} x} \, dx = \frac{\pi}{2^{2n+1}} \binom{2n}{n} \tag{3.7}$$

corresponds to the case $a = 0$, $b = -2n - 1$ and $c = n - \frac{1}{2}$.

Example 3.4. Formula **3.624.4** in [**2**]

$$\int_0^{\pi/4} \frac{\cos^{\mu}(2x)}{\cos^{2(\mu+1)} x} \, dx = 2^{2\mu} B(\mu+1, \mu+1) \tag{3.8}$$

corresponds to $a = 0$, $b = -2\mu - 2$ and $c = \mu$. Then (3.2) gives

$$\int_0^{\pi/4} \frac{\cos^{\mu}(2x)}{\cos^{2(\mu+1)} x} \, dx = \frac{1}{2} B\left(\frac{1}{2}, \mu+1 \right). \tag{3.9}$$

The duplication formula

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma(x + \tfrac{1}{2}), \tag{3.10}$$

transforms (3.9) into (3.8).

Example 3.5. The values $a = 2\mu - 2$, $b = 0$ and $c = \mu$ produce **3.624.5**:

$$\int_0^{\pi/4} \frac{\sin^{2\mu-2} x}{\cos^{\mu}(2x)}\,dx = \frac{\Gamma(\mu - \frac{1}{2})\Gamma(1-\mu)}{2\sqrt{\pi}} \tag{3.11}$$

directly. Indeed, the answer from (3.2) is $B(\mu - 1/2, 1-\mu)/2$. The table also has the alternative answer $2^{1-2\mu}B(2\mu-1, 1-\mu)$ that can be obtained using (3.10).

Example 3.6. Formula **3.625.1**:

$$\int_0^{\pi/4} \frac{\sin^{2n-1} x \, \cos^{p}(2x)}{\cos^{2p+2n+2} x}\,dx = \frac{1}{2}B(n, p+1) \tag{3.12}$$

corresponds to $a = 2n-1$, $b = -2p-2n-1$ and $c = p$.

Example 3.7. The choice $a = 2n-1$, $b = -2n-2m$ and $c = m - \frac{1}{2}$ gives **3.625.3**:

$$\int_0^{\pi/4} \frac{\sin^{2n-1} x \, \cos^{m-1/2}(2x)}{\cos^{2n+2m} x}\,dx = \frac{1}{2}B(n, m+\tfrac{1}{2}). \tag{3.13}$$

For n, $m \in \mathbb{N}$ we can also write

$$\int_0^{\pi/4} \frac{\sin^{2n-1} x \, \cos^{m-1/2}(2x)}{\cos^{2n+2m} x}\,dx = \frac{2^{2n-1}}{n}\binom{2m}{m}\binom{2n+2m}{n+m}^{-1}\binom{n+m}{n}^{-1}. \tag{3.14}$$

Example 3.8. The values $a = 2n$, $b = -2n-2m-1$ and $c = m - \frac{1}{2}$ give **3.625.4**:

$$\int_0^{\pi/4} \frac{\sin^{2n} x \, \cos^{m-1/2}(2x)}{\cos^{2n+2m+1} x}\,dx = \frac{1}{2}B\left(n+\tfrac{1}{2}, m+\tfrac{1}{2}\right). \tag{3.15}$$

For n, $m \in \mathbb{N}$ we can also write

$$\int_0^{\pi/4} \frac{\sin^{2n} x \, \cos^{m-1/2}(2x)}{\cos^{2n+2m+1} x}\,dx = \frac{\pi}{2^{2n+2m+1}}\binom{2n}{n}\binom{2m}{m}\binom{n+m}{n}^{-1}. \tag{3.16}$$

Example 3.9. Formula **3.626.1**:

$$\int_0^{\pi/4} \frac{\sin^{2n-1} x}{\cos^{2n+2} x}\sqrt{\cos(2x)}\,dx = \frac{1}{2}B(n, 3/2), \tag{3.17}$$

comes from (3.2) with $a = 2n-1$, $b = -2n-2$ and $c = 1/2$. For $n \in \mathbb{N}$ we have

$$\int_0^{\pi/4} \frac{\sin^{2n-1} x}{\cos^{2n+2} x}\sqrt{\cos(2x)}\,dx = \frac{2^{2n}(n-1)!\,n!}{(2n+1)!}. \tag{3.18}$$

Example 3.10. The last example in this section is formula **3.626.2**:

$$\int_0^{\pi/4} \frac{\sin^{2n} x}{\cos^{2n+3} x}\sqrt{\cos(2x)}\,dx = \frac{1}{2}B(n+\tfrac{1}{2}, \tfrac{3}{2}), \tag{3.19}$$

comes from (3.2) with $a = 2n$, $b = -2n-3$ and $c = 1/2$. For $n \in \mathbb{N}$ we have

$$\int_0^{\pi/4} \frac{\sin^{2n} x}{\cos^{2n+3} x}\sqrt{\cos(2x)}\,dx = \frac{\pi}{2^{2n+2}}\frac{(2n)!}{n!\,(n+1)!}. \tag{3.20}$$

References

[1] G. Boros and V. Moll. *Irresistible Integrals.* Cambridge University Press, New York, 1st edition, 2004.

[2] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products.* Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition, 2007.

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