THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 28: FRESNEL INTEGRALS. PRELIMINARY VERSION: LAST UPDATE OCTOBER 5, 2006.

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Abstract. We present a systematic derivation of some of the definite integrals in the classical table of Gradshteyn and Rhyzik that are related to the Fresnel integral.

1. Introduction

The table of integrals [1] contains some evaluations that can be related to the *Fresnel integrals*

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin t^2 \, dt, \tag{1.1}$$

and

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos t^2 \, dt. \tag{1.2}$$

• 8.250.2 and 8.250.3

Our goal is to present in a systematic manner, the evaluations appearing in the classical table of Gradshteyn and Rhyzik [1], that involve this function. These definitions appears as 8.250.2 and 8.250.3 respectively.

2. The complete Fresnel integrals

The *complete Fresnel* integrals are

$$\int_0^\infty \sin t^2 \, dt = \frac{1}{2}\sqrt{\frac{\pi}{2}} \tag{2.1}$$

and

$$\int_0^\infty \cos t^2 \, dt = \frac{1}{2}\sqrt{\frac{\pi}{2}} \tag{2.2}$$

The change of variables $t = \sqrt{a}x$, with $a > 0$ yield

$$\int_0^\infty \sin(ax^2) \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}} \tag{2.3}$$

and

$$\int_0^\infty \cos(ax^2) \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}. \tag{2.4}$$

• 3.691.1

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These appear as 3.691.1 in [1].

A proof using complex integration is presented in Section 3.

3. A complex proof

The evaluation of the complete Fresnel integral can be obtained from

$$\int_0^\infty e^{-i\lambda x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}e^{-i\pi/4}. \tag{3.1}$$

• 3.322.3

This appears as 3.322.3 in [1].

Consider the contour γ consisting of the real axis from 0 to R, going around the circle centered at 0 and radius R and then coming back to theorigin along the ray of angle $\pi/4$. Then integrating the function

$$f(z) = e^{-i\lambda z^2} \tag{3.2}$$

along γ, we have:

$$0 = \int_0^R e^{-i\lambda x^2}\,dx + \int_0^{\pi/4} \exp\left(-i\lambda(Re^{i\theta})^2 iRe^{i\theta}i\right)\,d\theta - \int_0^R \exp\left(-i\lambda t^2 e^{-\frac{\pi i}{2}}\right)e^{-i\pi/4}\,dt.$$

Now let $R \to \infty$ and observe that

$$\int_0^R e^{-i\lambda x^2}\,dx = e^{-\pi i/4}\int_0^R e^{-\lambda t^2}\,dt \to \frac{\sqrt{\pi}}{2\sqrt{\lambda}}e^{-i\pi/4}. \tag{3.3}$$

Now use

$$e^{\pi i/4} = \frac{\sqrt{2}}{2}(1-i), \tag{3.4}$$

and the fact that the integral over the arc tends to 0 as $R \to \infty$, to obtain the result.

Now consider the real and imaginary parts of the previous evaluation to obtain

$$\int_0^\infty \cos(\lambda x^2)\,dx = \int_0^\infty \sin(\lambda x^2)\,dx = \frac{\sqrt{2\pi}}{4\sqrt{\lambda}}. \tag{3.5}$$

The change of variables $t = \sqrt{\lambda}x$ produces

$$\int_0^\infty \cos(t^2)\,dt = \int_0^\infty \sin(t^2)\,dt = \frac{\sqrt{\pi}}{2\sqrt{2}}. \tag{3.6}$$

In terms of the functions C and S, introduced in (1.2) and (1.1), the complete Fresnel integrals state that

$$C(\infty) = S(\infty) = \frac{1}{2}. \tag{3.7}$$

4. Some elementary changes of variables

The change of variables $t = x^2$ in (2.3) yields 3.757.1: • 3.757.1

$$\int_0^\infty \frac{\sin(at)}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{2a}}. \tag{4.1}$$

• 3.757.2 The same change of variables converts (2.4) into 3.757.2:

$$\int_0^\infty \frac{\cos(at)}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{2a}}. \tag{4.2}$$

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References

[1] I.S. Gradshteyn and I.M. Rhyzik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.

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