THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 18: EVALUATIONS USING THE HURWITZ ZETA FUNCTION. PRELIMINARY VERSION: LAST UPDATE SEPTEMBER 22, 2006.

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Abstract. The table of Gradshteyn and Rhyzik contains many integrals that can be evaluated using the Hurwitz zeta function. Some examples are discussed.

1. Introduction

The Hurwitz zeta function is defined by

$$\zeta(s,q) = \sum_{k=1}^{\infty} \frac{1}{(k+q)^s}. \tag{1.1}$$

The series converges for $\text{Re}\, s > 1$.

2. A first integral representation

The integral representation

$$\zeta(s,q) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-qx}}{1-e^{-x}}\, dx \tag{2.1}$$

follows directly by expanding the integrand as

$$\frac{x^{s-1}e^{-qx}}{1-e^{-x}} = \sum_{k=0}^{\infty} x^{s-1}e^{-(q+k)x}. \tag{2.2}$$

Therefore

$$\int_0^\infty \frac{x^{s-1}e^{-qx}}{1-e^{-x}}\, dx = \sum_{k=0}^{\infty} \int_0^\infty x^{s-1}e^{-(q+k)x}\, dx. \tag{2.3}$$

The change of variables $t = (q+k)x$ and the integral representation of the gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\, dt \tag{2.4}$$

produce the result.

• 3.411.8 The table [1] contains as 3.411.8 the evaluation

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$$\int_0^\infty \frac{x^{s-1}\, e^{-qx}\, dx}{1+e^{x}} = \Gamma(s) \sum_{k=1}^\infty \frac{(-1)^{k-1}}{(q+k)^s}, \tag{2.5}$$

that we prefer to write as

$$\int_0^\infty \frac{x^{s-1}\, e^{-qx}\, dx}{1+e^{-x}} = \Gamma(s) \sum_{k=0}^\infty \frac{(-1)^{k}}{(q+k)^s}, \tag{2.6}$$

To confirm this, we expand the integrand as a geometric series

$$\frac{x^{s-1}e^{-qx}}{1+e^{-x}} = \sum_{k=0}^\infty (-1)^k x^{s-1} e^{-(q+k)x}.$$

The change of variables $u=(q+k)x$ produces

$$\int_0^\infty \frac{x^{s-1}\, e^{-qx}\, dx}{1+e^{-x}} = \sum_{k=0}^\infty \frac{(-1)^k}{(q+k)^s} \int_0^\infty u^{s-1}e^{-u}\, du. \tag{2.7}$$

The integral is $\Gamma(s)$ and we obtain the result.

Separating the terms according to the parity of k we get

$$\begin{aligned}\sum_{k=0}^\infty \frac{(-1)^k}{(q+k)^s} &= \sum_{j=0}^\infty \frac{1}{(q+2j)^s} - \sum_{j=0}^\infty \frac{1}{(q+2j+1)^s} \\ &= 2^{-s}\left(\sum_{j=0}^\infty \frac{1}{(j+q/2)^s} - \sum_{j=0}^\infty \frac{1}{(j+(q+1)/2)^s} \right).\end{aligned}$$

We conclude that

$$\int_0^\infty \frac{x^{s-1}\, e^{-qx}\, dx}{1+e^{-x}} = 2^{-s}\left(\zeta(s,q/2) - \zeta(s,(q+1)/2)\right). \tag{2.8}$$

We summarize this discussion in a theorem.

Theorem 2.1. *Assume the stated integrals converge. Then*

$$\int_0^\infty \frac{x^{s-1}e^{-qx}}{1-e^{-x}}\, dx = \Gamma(s)\zeta(s,q). \tag{2.9}$$

In particular, if $q \in \mathbb{N}$, we have

$$\int_0^\infty \frac{x^{s-1}\, e^{-qx}\, dx}{1-e^{-x}} = \Gamma(s)\left(\zeta(s) - \sum_{k=1}^{q-1} \frac{1}{k^s} \right). \tag{2.10}$$

Also

$$\begin{aligned}\int_0^\infty \frac{x^{s-1}\, e^{-qx}\, dx}{1+e^{-x}} &= 2^{-s}\Gamma(s)\,\left(\zeta(s,\tfrac{q}{2}) - \zeta(s,\tfrac{q+1}{2})\right) & (2.11)\\ &= \Gamma(s)\sum_{k=0}^\infty \frac{(-1)^k}{(k+q)^s}. & (2.12)\end{aligned}$$

In particular, if $q \in \mathbb{N}$, we have

$$\int_0^\infty \frac{x^{s-1}\, e^{-qx}\, dx}{1+e^{-x}} = (-1)^q\Gamma(s)\left(-2^{-s}(2^s-2)\zeta(s) + \sum_{k=1}^{q-1} \frac{(-1)^{k-1}}{k^s} \right). \tag{2.13}$$

Proof. The only part that needs to be verified is the statement when $q \in \mathbb{N}$. In that case

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+q)^s} &= (-1)^q \sum_{k=q}^{\infty} \frac{(-1)^k}{k^s} \\ &= (-1)^q \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} - \sum_{k=1}^{q-1} \frac{(-1)^k}{k^s} \right). \end{aligned}$$

The statement now follows from

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = (2^{1-s}-1)\zeta(s), \tag{2.14}$$

that comes from splitting the series according to the parity of k. □

We will now provide several special cases of the formulas described above that appear in [1]. Starting with the the change of variables $x = tb$ in (2.9) that yields
• 3.411.7 3.411.7:

$$\int_0^{\infty} \frac{x^{s-1}e^{-qx}}{1-e^{-bx}}\,dx = \Gamma(s) b^{-s} \zeta\left(s, \frac{q}{b}\right). \tag{2.15}$$

Special cases: we now describe several special cases of Theorem 2.1 that appear in [1].

• The evaluation (2.12) gives 3.411.8:

$$\int_0^{\infty} \frac{x^{n-1}e^{-qx}\,dx}{1+e^{x}} = \int_0^{\infty} \frac{x^{n-1}e^{-(q+1)x}\,dx}{1+e^{-x}} = (n-1)! \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k+q)^s},$$

• 3.411.8 We have replaced the parameter p in the table for q to be consistent with the notation in Theorem 2.1.

• 3.411.10 • The identity (2.13) with $q = 2$ and $s = 2$ yields the evaluation of 3.411.10:

$$\int_0^{\infty} \frac{xe^{-2x}\,dx}{1+e^{-x}} = 1 - \frac{\pi^2}{12}. \tag{2.16}$$

• 3.411.11 • The values $q = 3$ and $s = 2$ in (2.13) give the value of 3.411.11:

$$\int_0^{\infty} \frac{xe^{-3x}\,dx}{1+e^{-x}} = \frac{\pi^2}{12} - \frac{3}{4}. \tag{2.17}$$

• 3.411.12 • The choice $q = 2n+1$ and $s = 2$ in (2.13) give 3.411.12 in the form

$$\int_0^{\infty} \frac{xe^{-2nx}\,dx}{1+e^{x}} = \int_0^{\infty} \frac{xe^{-(2n+1)x}\,dx}{1+e^{-x}} = \frac{\pi^2}{12} - \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k^2}. \tag{2.18}$$

• 3.411.13 Similarly, $q = 2n$ and $s = 2$ give 3.411.13:

$$\int_0^{\infty} \frac{xe^{-(2n-1)x}\,dx}{1+e^{x}} = \int_0^{\infty} \frac{xe^{-2nx}\,dx}{1+e^{-x}} = -\frac{\pi^2}{12} + \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k^2}. \tag{2.19}$$

• 3.411.15 • The choice $q = n$ and $s = 3$ in (2.13) produces 3.411.15:

$$\int_0^\infty \frac{x^2 e^{-nx}\,dx}{1+e^{-x}} = (-1)^{n+1}\left(\frac{3}{2}\zeta(3) + 2\sum_{k=1}^{n-1}\frac{(-1)^k}{k^3}\right). \tag{2.20}$$

Similarly, $q = n$ and $s = 4$ in (2.13) gives 3.411.18: • 3.411.18

$$\int_0^\infty \frac{x^3 e^{-nx}\,dx}{1+e^{-x}} = (-1)^{n+1}\left(\frac{7\pi^4}{120} + 6\sum_{k=1}^{n-1}\frac{(-1)^k}{k^4}\right). \tag{2.21}$$

• The evaluation of 3.411.26:

$$\int_0^\infty xe^{-x}\frac{(1-e^{-x})}{1+e^{-3x}}\,dx = \frac{2\pi^2}{27} \tag{2.22}$$

requires a new trick. The integral is written as • 3.411.26

$$I = \int_0^\infty \frac{xe^{-x}\,dx}{1+e^{-3x}} - \int_0^\infty \frac{xe^{-2x}\,dx}{1+e^{-3x}}, \tag{2.23}$$

and the change of variables $t = 3x$ and (2.12) produces

$$\int_0^\infty xe^{-x}\frac{(1-e^{-x})}{1+e^{-3x}}\,dx = \sum_{k=0}^\infty \frac{(-1)^k}{(3k+1)^2} - \sum_{k=0}^\infty \frac{(-1)^k}{(3k+2)^2}. \tag{2.24}$$

To simplify these series observe that

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^2} = \sum_{k=1}^\infty \frac{(-1)^{3k}}{(3k)^2} + \sum_{k=0}^\infty \frac{(-1)^{3k+1}}{(3k+1)^2} + \sum_{k=0}^\infty \frac{(-1)^{3k+2}}{(3k+2)^2}.$$

We conclude that

$$\sum_{k=0}^\infty \frac{(-1)^k}{(3k+1)^2} - \sum_{k=0}^\infty \frac{(-1)^k}{(3k+2)^2} = -\frac{8}{9}\sum_{k=1}^\infty \frac{(-1)^k}{k^2}. \tag{2.25}$$

The result follows from

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}. \tag{2.26}$$

Integrals with the factor $1 - e^{-x}$ in the denominator are computed using (2.9) and (2.10).

• The evaluation of 3.411.9:

$$\int_0^\infty \frac{xe^{-x}\,dx}{e^x - 1} = \int_0^\infty \frac{xe^{-2x}\,dx}{1-e^{-x}} = \frac{\pi^2}{6} - 1 \tag{2.27}$$

comes from (2.10) with $q = 2$ and $s = 2$. • 3.411.9

• The value of 3.411.14:

$$\int_0^\infty \frac{x^2 e^{-nx}\,dx}{1-e^{-x}} = 2\left(\zeta(3) - \sum_{k=1}^{n-1}\frac{1}{k^3}\right), \tag{2.28}$$

comes from (2.10) with $q = n$ and $s = 3$. • 3.411.14

• Similarly, (2.10) with the parameters $q = n$ and $s = 4$ gives • 3.411.17

$$\int_0^\infty \frac{x^3 e^{-nx}\,dx}{1-e^{-x}} = \frac{\pi^2}{15} - 6\sum_{k=1}^{n-1} \frac{1}{k^4}. \tag{2.29}$$

• The evaluation of 3.411.21:

$$\int_0^\infty x^{n-1}\,\frac{1-e^{-mx}}{1-e^{x}}\,dx = -(n-1)!\sum_{k=1}^{m}\frac{1}{k^n}, \tag{2.30}$$

• 3.411.21 comes from (2.10) when it is writtes as

$$\begin{aligned} I &= -\int_0^\infty \frac{x^{n-1}e^{-x}\,dx}{1-e^{-x}} + \int_0^\infty \frac{x^{n-1}e^{-(m+1)x}\,dx}{1-e^{-x}} \\ &= -\Gamma(n)\zeta(n) + \Gamma(n)\left(\zeta(n) - \sum_{k=1}^{m}\frac{1}{k^n}\right). \end{aligned}$$

There is a sign missing in this entry in [1].

• The evaluation of 3.411.25:

$$\int_0^\infty x\frac{1+e^{-x}}{e^x-1}\,dx = \frac{\pi^2}{3} - 1, \tag{2.31}$$

• 3.411.25 comes from (2.10) when written as

$$\begin{aligned} \int_0^\infty x\frac{1+e^{-x}}{e^x-1}\,dx &= \int_0^\infty \frac{xe^{-x}}{1-e^{-x}}\,dx - \int_0^\infty \frac{xe^{-2x}}{1-e^{-x}}\,dx \\ &= \Gamma(2)\zeta(2) + \Gamma(2)(\zeta(2)-1) = \frac{\pi^2}{3} - 1. \end{aligned}$$

3. A logarithmic integral

We now present a version of Theorem 2.1 written in a logarithmic scale.

Theorem 3.1. *Assume the stated integrals converge. Then*

$$\int_0^1 \frac{t^{q-1}(\ln t)^{s-1}}{1-t}\,dt = (-1)^{s-1}\Gamma(s)\zeta(s,q). \tag{3.1}$$

In particular, if $q \in \mathbb{N}$*, we have*

$$\int_0^1 \frac{t^{q-1}\,(\ln t)^{s-1}\,dt}{1-t} = (-1)^{s-1}\Gamma(s)\left(\zeta(s) - \sum_{k=1}^{q-1}\frac{1}{k^s}\right). \tag{3.2}$$

Also

$$\int_0^1 \frac{t^{q-1}\,(\ln t)^{s-1}\,dt}{1+t} = (-1)^{s-1}2^{-s}\Gamma(s)\,\left(\zeta(s,\tfrac{q}{2}) - \zeta(s,\tfrac{q+1}{2})\right) \tag{3.3}$$

$$= (-1)^{s-1}\Gamma(s)\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+q)^s}. \tag{3.4}$$

In particular, if $q \in \mathbb{N}$*, we have*

$$\int_0^1 \frac{t^{q-1}\,(\ln t)^{s-1}\,dt}{1+t} = (-1)^{q+s-1}\Gamma(s)\left(-2^{-s}(2^s-2)\zeta(s) + \sum_{k=1}^{q-1}\frac{(-1)^{k-1}}{k^s}\right). \tag{3.5}$$

Proof. Let $t = e^{-x}$ in Theorem 2.1. □

We now present the evaluations in [1] that are consecuences of Theorem 3.1.

The integrals with denominator $1-t$ are presented first:

• The case $s=2$ gives 4.251.4: • 4.251.4

$$\int_0^1 \frac{t^{q-1}\,\ln t}{1-t}\,dt = -\zeta(2,q). \tag{3.6}$$

• The case $s=2$ and $q=2n$ yields

$$\int_0^1 \frac{t^{2n-1}\,\ln t\,dt}{1-t} = -\frac{\pi^2}{6} + \sum_{k=1}^{2n-1} \frac{1}{k^2}. \tag{3.7}$$

In particular, $n=1$ yields

$$\int_0^1 \frac{t\,\ln t\,dt}{1-t} = 1 - \frac{\pi^2}{6}. \tag{3.8}$$

This appears as 4.231.3. This value and the elementary integral • 4.231.3

$$\int_0^1 \frac{\ln x\,dx}{1-x} = -\frac{\pi^2}{6} \tag{3.9}$$

that appears as 4.231.2 give the value of 4.231.4: • 4.231.4

$$\int_0^1 \frac{1+x}{1-x}\,\ln x\,dx = 1 - \frac{\pi^2}{3}. \tag{3.10}$$

• The choice $s=3$ and $q=n+1$ in (3.2) produces 4.261.12: • 4.261.12

$$\int_0^1 \frac{t^n\,(\ln t)^2\,dt}{1-t} = 2\left(\zeta(3) - \sum_{k=1}^{n} \frac{1}{k^3}\right). \tag{3.11}$$

• The change of variables $t=x^2$ in entry 4.261.13 yields

$$\int_0^1 \frac{x^{2n}\,\ln^2 x}{1-x^2}\,dx = \frac{1}{8}\int_0^1 \frac{t^{n-1/2}\,\ln^2 t}{1-t}\,dt. \tag{3.12}$$

This corresponds to the choice $s=3$ and $q=n+1/2$ in (3.2):

$$\begin{aligned}\int_0^1 \frac{t^{n-1/2}\,(\ln t)^2\,dt}{1-t} &= 2\sum_{k=0}^{\infty} \frac{1}{(n+1/2+k)^3} \\ &= 16\sum_{k=0}^{\infty} \frac{1}{(2n+1+2k)^3} \\ &= 16\sum_{k=n}^{\infty} \frac{1}{(2k+1)^3}.\end{aligned}$$

Therefore • 4.261.13

$$\int_0^1 \frac{x^{2n}\,\ln^2 x}{1-x^2}\,dx = 2\sum_{k=n}^{\infty} \frac{1}{(2k+1)^3}.$$

• The choice $s=4$ and $q=n+1$ in (3.2) produces 4.262.5: • 4.262.5

$$\int_0^1 \frac{t^n (\ln t)^3 \, dt}{1-t} = -\frac{\pi^4}{15} + 6\sum_{k=1}^{n} \frac{1}{k^4}. \tag{3.13}$$

• The choice $q = 1$ and $s = 4$ give 4.262.2:

$$\int_0^1 \frac{(\ln t)^3 \, dt}{1-t} = -\frac{\pi^4}{15}, \tag{3.14}$$

• 4.262.2 using the value $\zeta(4) = \pi^4/90$.

We now present the integrals with denominator $1+t$:

• The case $q = 2n$ and $s = 2$ in (3.5) yields

$$\int_0^1 \frac{t^{2n-1} \ln t}{1+t} dt = \frac{\pi^2}{12} - \sum_{k=1}^{2n-1} \frac{(-1)^k}{k^2}. \tag{3.15}$$

• 4.251.6 This appears as 4.251.6.

• The case $q = 2n+1$ and $s = 2$ in (3.5) yields

$$\int_0^1 \frac{t^{2n} \ln t}{1+t} dt = -\frac{\pi^2}{12} - \sum_{k=1}^{2n} \frac{(-1)^k}{k^2}. \tag{3.16}$$

• 4.251.5 This appears as 4.251.5.

• The case $q = n+1$ and $s = 3$ gives from (3.5) the evaluation of 4.261.11:

$$\int_0^1 \frac{t^n (\ln t)^2}{1+t} \, dt = (-1)^n \left(\frac{3}{2}\zeta(3) + 2\sum_{k=1}^{n} \frac{(-1)^k}{k^3} \right). \tag{3.17}$$

• 4.261.11 This answer is equivalent to the one given in [1]:

$$\int_0^1 \frac{t^n (\ln t)^2}{1+t} \, dt = 2\sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{(k+1)^3}. \tag{3.18}$$

• The case $q = 1$ and $s = 4$ yields

$$\int_0^1 \frac{(\ln t)^3 \, dt}{1+t} = -\frac{7\pi^4}{120}. \tag{3.19}$$

• 4.262.1 This appears as 4.262.1 in [1].

4. Integrals over the whole line

The result of Theorem 2.1 provides the evaluation of 3.411.16:

$$\int_{-\infty}^{\infty} \frac{x^2 e^{-ax}}{1+e^{-x}} \, dx = \frac{\pi^3 \left(2 - \sin^2 \pi a\right)}{\sin^3 \pi a}. \tag{4.1}$$

• 3.411.16 To check this, we write $I = I_1 + I_2$, where

$$I_1 = \int_0^{\infty} \frac{x^2 e^{-ax}}{1+e^{-x}} \, dx \tag{4.2}$$

and, after the change of variables $x \mapsto -x$,

$$I_2 = \int_0^\infty \frac{x^2 e^{ax}}{1+e^x}\, dx. \tag{4.3}$$

The value of I_1 is given by

$$I_1 = 2 \sum_{k=0}^\infty \frac{(-1)^k}{(k+a)^3}, \tag{4.4}$$

and writing

$$I_2 = \int_0^\infty \frac{x^2 e^{(a-1)x}}{1+e^{-x}}\, dx \tag{4.5}$$

we obtain

$$I_2 = 2 \sum_{k=0}^\infty \frac{(-1)^k}{(k+1-a)^3}. \tag{4.6}$$

We conclude that

$$\int_{-\infty}^\infty \frac{x^2 e^{-ax}}{1+e^{-x}}\, dx = 2 \sum_{k=-\infty}^\infty \frac{(-1)^k}{(k+a)^3}. \tag{4.7}$$

To prove the stated result, start with the classical

$$\sum_{k=-\infty}^\infty \frac{(-1)^k}{k+a} = \frac{\pi}{\sin(\pi a)}, \tag{4.8}$$

and differentiate twice with respect to the parameter a.

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References

[1] I.S. Gradshteyn and I.M. Rhyzik. *Table of Integrals, Series, and Products.* Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.

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