THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 20: HYPERBOLIC FUNCTIONS. PRELIMINARY VERSION: LAST UPDATE SEPTEMBER 18, 2006.

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Abstract. We present a systematic derivation of some of the definite integrals in the classical table of Gradshteyn and Rhyzik involving hyperbolic functions.

1. Introduction

The table of integrals [1] contains some evaluations that can be derived from the standard *hyperbolic functions*, defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}, \tag{1.1}$$

and

$$\cosh x = \frac{e^x + e^{-x}}{2}. \tag{1.2}$$

Our goal is to present in a systematic manner, the evaluations appearing in the classical table of Gradshteyn and Rhyzik [1], that involve these functions.

2. Some elementary examples

In the evaluation of 3.511.1 in [1]:

$$\int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2a}, \quad \text{for } a > 0, \tag{2.1}$$

• 3.511.1

we see that the parameter a can be scaled out of the equation. Indeed, the change of variables $t = ax$ yields

$$\int_0^\infty \frac{dt}{\cosh t} = \frac{\pi}{2}. \tag{2.2}$$

This can be reduced to a rational integrand by the change of variables $s = e^t$. We obtain

$$\begin{aligned} \int_0^\infty \frac{dt}{\cosh t} &= 2\int_1^\infty \frac{ds}{s^2+1} \\ &= 2\left(\tan^{-1}(\infty) - \tan^{-1}\frac{\pi}{4}\right) = \frac{\pi}{2}. \end{aligned}$$

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Actually, the change of variables $s = e^t$ produces the value of the indefinite integral:

$$\int \frac{dt}{\cosh t} = 2 \int \frac{ds}{s^2+1} \tag{2.3}$$

and we obtain

$$\int \frac{dt}{\cosh t} = 2 \tan^{-1}(e^t). \tag{2.4}$$

• 2.423.9

This appears as 2.423.9.

3. An example that is evaluated in terms of the Hurwitz zeta function

Special cases of the evaluation

$$\int_0^\infty \frac{x^n \, dx}{\cosh(x^m)} = \frac{\Gamma(p)}{m\, 2^{2p-1}} \left[\zeta\left(p, \tfrac{1}{4}\right) - \zeta\left(p, \tfrac{3}{4}\right) \right], \tag{3.1}$$

appear in [1]. Here $p = \frac{n+1}{m}$ and

$$\zeta(z,q) = \sum_{k=1}^\infty \frac{1}{(k+q)^z} \tag{3.2}$$

is the Hurwitz zeta function. To prove (3.1) simply write

$$\int_0^\infty \frac{x^n \, dx}{\cosh(x^m)} = 2 \int_0^\infty \frac{x^n e^{-x^m} \, dx}{1+e^{-2x^m}}. \tag{3.3}$$

Now expand the integrand as a geometric series and produce

$$\begin{aligned} I &= 2 \sum_{j=0}^\infty (-1)^j \int_0^\infty x^n e^{-(2j+1)x^m} \, dx \\ &= 2 \sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)^p} \int_0^\infty t^n e^{-t^m} \, dt. \end{aligned}$$

The change of variables $u = t^m$ shows that

$$\begin{aligned} \int_0^\infty t^n e^{-t^m} \, dt &= \frac{1}{m} \int_0^\infty u^{p-1} e^{-u} \, du \\ &= \frac{1}{m} \Gamma(p). \end{aligned}$$

We conclude that

$$I = \frac{2\Gamma(p)}{m} \sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)^p}. \tag{3.4}$$

Now split the sum according to the parity of j:

$$\begin{aligned} \sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)^p} &= \sum_{j=0}^\infty \frac{1}{(4j+1)^p} - \sum_{j=0}^\infty \frac{1}{(4j+3)^p} \\ &= 2^{-2p} \left(\zeta(p, \tfrac{1}{4}) - \zeta(p, \tfrac{3}{4}) \right). \end{aligned}$$

Thus, we have

$$\int_0^\infty \frac{x^n\,dx}{\cosh(x^m)} = \frac{\Gamma(p)}{m\,2^{2p-1}}\left[\zeta\left(p,\tfrac{1}{4}\right) - \zeta\left(p,\tfrac{3}{4}\right)\right] = \frac{2\Gamma(p)}{m}\sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)^p}. \tag{3.5}$$

Example 1: in the case $n = m = 1$, we have $p = 2$ and we obtain 3.521.2:

$$\int_0^\infty \frac{x\,dx}{\cosh x} = 2G \tag{3.6}$$

• 3.521.2

where G is *Catalan's constant* defined by

$$G := \sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)^2}. \tag{3.7}$$

• 4.231.12

The change of variables $u = e^{-t}$ yields 4.231.12:

$$\int_0^1 \frac{\ln u\,du}{1+u^2} = -G. \tag{3.8}$$

Example 2: the case $n = 0$, $m = 2$ yields $p = 1/2$ and we obtain 3.511.8:

$$\int_0^\infty \frac{dx}{\cosh(x^2)} = \sqrt{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}, \tag{3.9}$$

• 3.511.8

where we have used $\Gamma(1/2) = \sqrt{\pi}$.

Example 3: the case $n = -1/2$, $m = 1$ yields $p = 1/2$ and we obtain 3.523.12.:

$$\int_0^\infty \frac{dx}{\sqrt{x}\,\cosh x} = 2\sqrt{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}, \tag{3.10}$$

• 3.523.12

Example 4: the case $n = 1/2$, $m = 1$ yields $p = 3/2$ and now we obtain 3.523.11.:

$$\int_0^\infty \frac{\sqrt{x}\,dx}{\cosh x} = \sqrt{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{(2k+1)^3}}, \tag{3.11}$$

• 3.523.11

where we have used $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$.

The evaluation of

$$\int_0^\infty \frac{x^n\,dx}{\sinh(x^m)} = \frac{2\Gamma(p)}{m}\sum_{j=0}^\infty \frac{1}{(2j+1)^p}, \tag{3.12}$$

with $p = (n+1)/m$ is done exactly as above. Using the identity

$$\sum_{j=0}^\infty \frac{1}{(2j+1)^p} = \frac{2^p-1}{2^p}\sum_{j=0}^\infty \frac{1}{j^p} \tag{3.13}$$

produces

$$\int_0^\infty \frac{x^n\,dx}{\sinh(x^m)} = \frac{\Gamma(p)}{m}\frac{2^p-1}{2^{p-1}}\zeta(p). \tag{3.14}$$

The special case $m = 1$, so that $p = n + 1$ yields

$$\int_0^\infty \frac{x^n\,dx}{\sinh x} = \Gamma(n+1)\frac{2^{n+1}-1}{2^n}\zeta(n+1). \tag{3.15}$$

This appears as 3.523.1 in [1]. In particular, when $n = 1$ we obtain 3.521.1: • 3.523.1

$$\int_0^\infty \frac{x\,dx}{\sinh x} = \frac{\pi^2}{4}. \tag{3.16}$$

This appears in the apparently more general form • 3.521.1

$$\int_0^\infty \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{4a^2}. \tag{3.17}$$

But this reduces to the case $a = 1$ by the change of variables $t = ax$.

We now consider the case $n = 2k - 1$ to obtain 3.253.2:

$$\int_0^\infty \frac{x^{2k-1}\,dx}{\sinh x} = \frac{2^{2k}-1}{2k}|B_{2k}|\pi^{2k}. \tag{3.18}$$

Here we have used • 3.523.2

$$\zeta(2k) = \frac{2^{2k-1}|B_{2k}|}{(2k)!}\pi^{2k}. \tag{3.19}$$

Using the values $B_4 = -1/30$, $B_6 = 1/42$ and $B_8 = 1/30$ we obtain 3.523.6: • 3.523.6

$$\int_0^\infty \frac{x^3\,dx}{\sinh x} = \frac{\pi^4}{8}, \tag{3.20}$$

and 3.523.8: • 3.523.8

$$\int_0^\infty \frac{x^5\,dx}{\sinh x} = \frac{\pi^6}{4}, \tag{3.21}$$

and 3.523.10: • 3.523.10

$$\int_0^\infty \frac{x^7\,dx}{\sinh x} = \frac{17\,\pi^8}{16}. \tag{3.22}$$

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References

[1] I.S. Gradshteyn and I.M. Rhyzik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.

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