THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 12: EVALUATION USING THE HYPERGEOMETRIC FUNCTION. PRELIMINARY VERSION: LAST UPDATE SEPTEMBER 8, 2006.

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Abstract. The table of Gradshteyn and Rhyzik contains many integrals that can be evaluated using the hypergeometric function. Some examples are discussed.

1. Introduction

The hypergeometric function is defined by

$$ {}_pF_q\left(a_1,\, a_2, \cdots, a_p; b_1,\, b_2, \cdots, b_q; x\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!} \tag{1.1}$$

Proposition 1.1. The hypergeometric function ${}_2F_1$ is given by

$$ {}_2F_1\left(a,\, b;\, c;\, x\right) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\, dt. \tag{1.2}$$

Here

$$ B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt \tag{1.3}$$

is the *beta* function.

Proof. Start □

• 3.197.3

This representation appears as 3.197.3 in [1]. Naturally here we will use it in the form

$$ \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\, dt = B(b, c-b)\, {}_2F_1\left(a,\, b;\, c;\, x\right) \tag{1.4}$$

the integral being the object of primary interest.

The special case $a = c = 1$ appears as 3.197.10:

$$ \int_0^1 \frac{t^{b-1}\, dt}{(1-t)^b\, (1+tx)} = \frac{\pi}{\sin(\pi b)} (1+x)^{-b}. \tag{1.5}$$

• 3.197.10

The evaluation is direct. Simply observe that

$$ {}_2F_1(1,\, b;\, 1;\, -x) = (1+x)^{-b}. \tag{1.6}$$

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Introduce now the index r by $r = a - b$ and take $c = b + r$. Then we have

$$\int_0^1 t^{b-1}(1-t)^{r-1}(1-tx)^{-b-r}\,dt = B(b,r)\,{}_2F_1\,(b+r,\,b;\,b+r;\,x) \tag{1.7}$$

The series representation shows that

$${}_2F_1\,(a,\,b;\,b;\,x) = (1-x)^{-a}, \tag{1.8}$$

so the previous evaluation reduces to

$$\int_0^1 t^{b-1}(1-t)^{r-1}(1-tx)^{-b-r}\,dt = B(b,r)\,(1-x)^{-b-r}. \tag{1.9}$$

• 3.197.4

This appears as 3.197.4 in [1].

2. Some examples

The hypergeometric function appears in the evaluation of integrals of the form

$$I = \int_a^b L_1(x)^{\mu-1}L_2(x)^{\nu-1}L_3(x)^{\lambda-1}\,dx \tag{2.1}$$

where L_j are linear functions and $L_1(a) = L_2(b) = 0$. For example, 3.198:

• 3.198

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\left[ax+b(1-x)+c\right]^{-(\mu+\nu)}\,dx = (a+c)^{-\mu}(b+c)^{-\nu}B(\mu,\nu) \tag{2.2}$$

is reduced to the normal form (1.2) by writing

$$I = (b+c)^{-\mu-\nu}\int_0^1 t^{\mu-1}(1-t)^{\nu-1}(1-rt)^{-(\mu+\nu)}\,dt \tag{2.3}$$

with $r = (b-a)/(b+c)$. Then (1.2) gives

$$I = (b+c)^{-\mu-\nu}B(\mu,\nu)_2F_1\left(\mu+\nu,\,\mu;\,\mu+\nu;\frac{b-a}{b+c}\right). \tag{2.4}$$

To produce the stated answer, simply observe the special value of the hypergeometric function

$${}_2F_1(a,\,b;\,a;z) = (1-z)^{-b}. \tag{2.5}$$

Similarly, the evaluation of 3.199:

$$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}(x-c)^{-\mu-\nu}\,dx = (b-a)^{\mu+\nu-1}(b-c)^{-\mu}(a-c)^{-\nu}B(\mu,\nu), \tag{2.6}$$

• 3.199

is reduced to the interval $[0,1]$ by $t = (x-a)/(b-a)$ and then the result follows from 3.198.

The specific form of the answer is sometimes simplified due to a special relation of the parameters μ, ν and λ in (2.1). For example, in the evaluation of 3.197.11:

$$\int_0^1 \frac{x^{p-1/2}\,dx}{(1-x)^p\,(1+qx)^p} = \frac{2}{\sqrt{\pi}}\Gamma\left(p+\tfrac{1}{2}\right)\Gamma(1-p)\,\cos^{2p}(\varphi)\frac{\sin((2p-1)\varphi)}{(2p-1)\sin(\varphi)}, \tag{2.7}$$

• 3.197.11

with $\varphi = \arctan(\sqrt{q})$. The standard reduction of the integral to hypergeometric form is easy. Write

$$I = x^{p-1/2}(1-x)^{-1}(1+qx)^{-p}\,dx \tag{2.8}$$

and use (1.2) to obtain

$$I = B(p+\tfrac{1}{2}, 1-p)\,{}_2F_1\left(p,\, p+\tfrac{1}{2};\, \tfrac{3}{2};\, -q\right). \tag{2.9}$$

To reduce the answer to the stated form, we employ 9.121.19:

$${}_2F_1\left(\frac{n+2}{2},\, \frac{n+1}{2};\, \frac{3}{2};\, -\tan^2 z\right) = \frac{\sin nz\, \cos^{n+1} z}{n\, \sin z}.$$

The evaluation of 3.197.12:

$$\int_0^1 \frac{x^{p-1/2}\,dx}{(1-x)^p\,(1-qx)^p} = \frac{\Gamma(p+\frac{1}{2})\Gamma(1-p)}{\sqrt{\pi}}\,\frac{\left[(1-\sqrt{q})^{1-2p} - (1+2\sqrt{q})^{1-2q}\right]}{(2p-1)\,\sqrt{q}}. \tag{2.10}$$

• 3.197.12

is done in similar form. The reduction to

$$I = B(p+\tfrac{1}{2}, 1-p){}_2F_1\left(p,\, p+\tfrac{1}{2};\, \tfrac{3}{2};\, q\right) \tag{2.11}$$

is direct from (1.2). The stated form now follows from 9.121.4:

$${}_2F_1\left(-\tfrac{n-1}{2}, -\tfrac{n}{2}+1;\, \tfrac{3}{2};\, \tfrac{z^2}{t^2}\right) = \frac{(t+z)^n - (t-z)^n}{2nzt^{n-1}}.$$

3. A FIRST SCALING

The change of variables $y = tu$ produces

$$\int_0^u y^{b-1}(u-y)^{c-b-1}(u-xy)^{-a}\,dy = u^{c-a-1}B(b, c-b){}_2F_1\left(a,\, b;\, c; x\right). \tag{3.1}$$

The special case $c = b+1$ produces

$$\int_0^u y^{b-1}(u-xy)^{-a}\,dy = \frac{1}{b}u^{b-a}{}_2F_1\left(a,\, b;\, b+1; x\right), \tag{3.2}$$

where we have used $B(b,1) = 1/b$. In order to eliminate the factor u^{-a}, we choose $x = -ur$ to obtain

$$\int_0^u y^{b-1}(1+ry)^{-a}\,dy = \frac{1}{b}u^b{}_2F_1\left(a,\, b;\, b+1; -ru\right), \tag{3.3}$$

• 3.194.1 This appears as 3.194.1 in [1]. The special case $a = 1$, stating that

$$\int_0^u \frac{y^{b-1}\,dy}{1+ry} = \frac{1}{b}u^b{}_2F_1\left(1,\, b;\, b+1; -ru\right), \tag{3.4}$$

• 3.194.5

appears as 3.194.5 in [1].

We now write (3.3) as an integral over an infinite half-line. Start with the change of variables $w = 1/y$

$$\int_{1/u}^{\infty} w^{a-b-1}(1+w/r)^{-a}\,dw = \frac{u^b r^a}{b}{}_2F_1\left(a,\, b;\, b+1; -ru\right), \tag{3.5}$$

Now replace u by $1/u$ and r by $1/r$ to produce

$$\int_u^\infty w^{a-b-1}(1+rw)^{-a}\,dw = \frac{1}{bu^br^a}\,{}_2F_1\left(a,\,b;\,b+1;-\frac{1}{ru}\right). \tag{3.6}$$

Now let $b = a-s$ to obtain

$$\int_u^\infty w^{s-1}(1+rw)^{-a}\,dw = \frac{1}{(a-s)u^{a-s}r^a}\,{}_2F_1\left(a,\,a-s;\,a-s+1;-\frac{1}{ru}\right). \tag{3.7}$$

This appears as 3.194.2 in [1]. • 3.194.2

4. Integrals over a half-line

The change of variable $y = 1/t$ converts (1.2) into 3.197.6:

$$\int_1^\infty y^{a-c}(y-1)^{c-b-1}(\alpha y-1)^{-a}\,dy = \alpha^{-a}B(b,c-b)\,{}_2F_1\left(a,\,b;\,c;1/\alpha\right) \tag{4.1}$$

• 3.197.6

where we have labelled $\alpha = 1/x$.

The change of variables $y = t/(1-t)$ converts (1.2) into 3.197.5:

$$\int_0^\infty y^{b-1}(1+y)^{a-c}(1+\alpha y)^{-a}\,dy = B(b,c-b)\,{}_2F_1\left(a,\,b;\,c;1-\alpha\right) \tag{4.2}$$

• 3.197.5

where we have labelled $\alpha = 1-x$. If we now replace α by 1α we obtain

$$\int_0^\infty y^{b-1}(1+y)^{a-c}(y+\alpha)^{-a}\,dy = \alpha^aB(b,c-b){}_2F_1\left(a,\,b;\,c;\,1-1/\alpha\right). \tag{4.3}$$

Now use the identity

$${}_2F_1\left(a,\,b;\,c;\,1-1/\alpha\right) = (1-\alpha)^a\,{}_2F_1\left(a,\,c-b;\,c;\,\alpha\right) \tag{4.4}$$

to produce 3.197.9:

$$\int_0^\infty y^{b-1}(1+y)^{a-c}(y+\alpha)^{-a}\,dy = \alpha^aB(b,c-b){}_2F_1\left(a,\,c-b;\,c;\,1-\alpha\right). \tag{4.5}$$

• 3.197.9

The change of variables $y = tu$ converts (1.2), with $-x$ instead of x, into 3.197.8:

$$\int_0^u y^{b-1}(u-y)^{c-b-1}(y+\alpha)^{-a}\,dy = \alpha^{-a}u^{c-1}B(b,c-b)\,{}_2F_1\left(a,\,b;\,c;-u/\alpha\right) \tag{4.6}$$

• 3.197.8

where we have labelled $\alpha = u/x$.

The change of variables $y = st/(1-t)$ converts (1.2) into

$$\int_0^\infty y^{b-1}(y+s)^{a-c}(y+r)^{-a}\,dy = r^{-a}s^{a+b-c}B(b,c-b){}_2F_1\left(a,\,b;\,c;\,1-\frac{s}{r}\right), \tag{4.7}$$

• 3.197.1

where $r = s/(1-x)$. This is 3.197.1 in [1]. The special case $a = c-1$ produces 3.227.1:

$$\int_0^\infty \frac{y^{b-1}(y+r)^{1-c}}{y+s}\,dy = r^{1-c}s^{b-1}B(b,c-b){}_2F_1\left(c-1,\,b;\,c;\,1-\frac{s}{r}\right), \tag{4.8}$$

• 3.227.1

We now shift the lower limit of integration via $x = y + u$ to produce

$$\int_u^\infty (x-u)^{b-1}(x-u+s)^{a-c}(x-u+r)^{-a}\,dx = r^{-a}u^{a+b-c}B(b,c-b)\,{}_2F_1\left(a,\, b;\, c;\, 1-\frac{s}{r}\right),$$

Now choose $s = u$ and introduce the parameter v by $v = r - u$ to get

$$\int_u^\infty x^{a-c}(x-u)^{b-1}(x+v)^{-a}\,dx = (v+u)^{-a}u^{a+b-c}B(b,c-b)\,{}_2F_1\left(a,\, b;\, c;\, \frac{v}{v+u}\right).$$

Introduce new parameters via $a = -p$, keeping b and $c = q - p$. Then we obtain

$$\begin{aligned}\int_u^\infty x^{-q}(x-u)^{b-1}(x+v)^{p}\,dx &= (v+u)^{p}u^{b-q}B(b,c-b-p)\,{}_2F_1\left(-p,\, b;\, q-p;\, \frac{v}{v+u}\right)\\ &= (v+u)^{p}u^{b-q}B(b,c-b-p)\,{}_2F_1\left(b,\, -p;\, q-p;\, \frac{v}{v+u}\right)\end{aligned}$$

where we have used the symmetry of the hypergeometric function in its two variables.

This evaluation can be written in a different form, using 9.131.1:

$$(4.9)\qquad {}_2F_1\left(a,\, b;\, c;\, z\right) = (1-z)_2^{-a}F_1\left(a,\, c-b;\, c;\, z/(z-1)\right).$$

We obtain

$$\int_u^\infty x^{-q}(x-u)^{b-1}(x+v)^{p}\,dx = (v+u)^{b+p}u^{b-q}B(b,q-p-b)\,{}_2F_1\left(b,\, q;\, q-p;\, -\frac{v}{u}\right).$$

• 3.197.2

This is the form that is found in 3.197.2.

5. A SECOND EXAMPLE

The table [1] contains the formula 3.196.1:

$$(5.1)\qquad \int_0^u (x+\beta)^\nu (u-x)^{\mu-1}\,dx = \frac{\beta^\nu\, u^\mu}{\mu}\,{}_2F_1\left[1, -\nu, 1+\mu, -\frac{u}{\beta}\right].$$

• 3.196.1 We believe that it is a bad idea to have u and μ in the same formula, so we write this as

$$(5.2)\qquad \int_0^v (x+\beta)^\nu (v-x)^{\mu-1}\,dx = \frac{\beta^\nu\, v^\mu}{\mu}\,{}_2F_1\left[1, -\nu, 1+\mu, -\frac{v}{\beta}\right].$$

To prove this, we let $x = vt$ to get

$$(5.3)\qquad \int_0^v (x+\beta)^\nu (v-x)^{\mu-1}\,dx = \beta^\nu v^\mu \int_0^1 (1+vt/\beta)^\nu (1-t)^{\mu-1}\,dt.$$

Using the integral representation (1.2) we obtain the result.

6. An exponential scale

The change of variables $t = e^{-r}$ in (1.2) produces

$$ {}_2F_1\left(a,\, b;\, c;\, x\right) = \frac{1}{B(b,c-b)} \int_0^\infty e^{-br}(1-e^{-r})^{c-b-1}(1-xe^{-r})^{-a}\, dr. \tag{6.1}$$

We now relabel the parameters by $a = \rho$, $b = \mu$, $c = \nu+\mu$, $x = \beta$ to produce 3.312.3:

• 3.312.3

$$\int_0^\infty (1-e^{-x})^{\nu-1}(1-\beta e^{-x})^{-\rho} e^{-\mu x}\, dx = B(\mu,\nu)\, {}_2F_1\left(\rho,\, \mu;\, \mu+\nu;\, \beta\right). \tag{6.2}$$

7. A more challenging example

The evaluation of 3.197.7

$$\int_0^\infty x^{\mu-1/2}(x+s)^{-\mu}(x+r)^{-\mu}\, dx = \sqrt{\pi}(\sqrt{r}+\sqrt{s})^{1-2\mu}\frac{\Gamma(\mu-1/2)}{\Gamma(\mu)} \tag{7.1}$$

• 3.197.7

requires some more properties of the hypergeometric function.

The scaling $x = rt$ produces

$$I = s^{-\mu}\sqrt{r}\int_0^\infty t^{\mu-1/2}(1+t)^{-\mu}(1+rt/s)^{\mu}\, dt \tag{7.2}$$

and using 3.197.5 we have

$$I = s^{-\mu}\sqrt{r} B\left(\mu+\tfrac{1}{2}, \mu-\tfrac{1}{2}\right)\, {}_2F_1\left(\mu, \mu+\tfrac{1}{2}, 2\mu;\, z\right) \tag{7.3}$$

where $z = 1 - r/s$. To simplify this expression we employ the relation

$$\begin{aligned} {}_2F_1(\alpha,\beta;\gamma;z) &= \frac{(1-z)^{-\alpha}\,\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}\, {}_2F_1(\alpha,\gamma-\beta;\alpha-\beta+1;\frac{1}{1-z}) + \\ &+ \frac{(1-z)^{-\beta}\,\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\, {}_2F_1(\beta,\gamma-\alpha;\beta-\alpha+1;\frac{1}{1-z}), \end{aligned}$$

to produce

$$\begin{aligned} {}_2F_1\left(\mu,\mu+\tfrac{1}{2},2\mu;\, z\right) &= \frac{(1-z)^{-\mu}\Gamma(2\mu)\Gamma(1/2)}{\Gamma(\mu+1/2)\,\Gamma(\mu)}\, {}_2F_1\left(\mu,\mu-\tfrac{1}{2}\,\tfrac{1}{2};\, \frac{1}{1-z}\right) \\ &+ \frac{(1-z)^{-\mu-1/2}\Gamma(2\mu)\Gamma(-1/2)}{\Gamma(\mu-1/2)\,\Gamma(\mu)}\, {}_2F_1\left(\mu,\mu+\tfrac{1}{2}\,\tfrac{3}{2};\, \frac{1}{1-z}\right). \end{aligned}$$

The binomial theorem shows that

$${}_2F_1\left(-\frac{n}{2}, -\frac{n-1}{2};\, \frac{1}{2};\, \frac{z^2}{t^2}\right) = \frac{1}{2t^n}\left((t+z)^n + (t-z)^n\right), \tag{7.4}$$

that appears as 9.121.2, so that

$${}_2F_1\left(\mu,\, \mu-\tfrac{1}{2};\, \tfrac{1}{2};\, \frac{1}{1-z}\right) = \frac{1}{2(1-z)^{1/2-\mu}}\left((1+\sqrt{1-z})^{1-2\mu} + (-1+\sqrt{1-z})^{1-2\mu}\right).$$

Similarly, 9.121.4 states that

$${}_2F_1\left(-\frac{n-1}{2}, -\frac{n-2}{2};\, \frac{3}{2};\, \frac{z^2}{t^2}\right) = \frac{1}{2nzt^{n-1}}\left((t+z)^n - (t-z)^n\right), \tag{7.5}$$

to produce

$$ {}_2F_1\left(\mu,\, \mu-\tfrac{1}{2};\, \tfrac{3}{2};\, \frac{1}{1-z}\right) = \frac{1}{2(1-2\mu)(1-z)^{-\mu}} \left((1+\sqrt{1-z})^{1-2\mu} - (-1+\sqrt{1-z})^{1-2\mu}\right). $$

Replacing these values in (7.3) produces the result.

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