THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 15: THE INCOMPLETE GAMMA FUNCTION. PRELIMINARY VERSION: LAST UPDATE SEPTEMBER 19, 2006.

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Abstract. We present a systematic derivation of some of the definite integrals in the classical table of Gradshteyn and Rhyzik that can be reduced to the incomplete beta function.

1. Introduction

The table of integrals [1] contains some evaluations that can be derived by elementary means from the *incomplete gamma function*, defined by

$$\gamma(a,x) := \int_0^x t^{a-1} e^{-t} \, dt \tag{1.1}$$

and a second one by

$$\Gamma(a,x) := \int_x^\infty t^{a-1} e^{-t} \, dt \tag{1.2}$$

• 8.350 These definitions appear as 8.350 in [1].

Our goal is to present in a systematic manner, the evaluations appearing in the classical table of Gradshteyn and Rhyzik [1], that involve these function.

2. Some elementary changes of variables

The change of variables $t = by$ produces

$$\int_0^{x/b} y^{a-1} e^{-by} \, dy = b^{-a} \gamma(a,x). \tag{2.1}$$

• 3.381.1

Now replace x by bx to obtain 3.381.1:

$$\int_0^x y^{a-1} e^{-by} \, dy = b^{-a} \gamma(a,bx). \tag{2.2}$$

Exactly the same argument produces 3.381.3:

$$\int_x^\infty y^{a-1} e^{-by} \, dy = b^{-a} \Gamma(a,bx). \tag{2.3}$$

• 3.381.3

Date: September 25, 2006.
1991 *Mathematics Subject Classification.* Primary 33.
Key words and phrases. Integrals.

The evaluation of 3.382.1:

$$\int_0^u (u-x)^a e^{-bx}\,dx = (-b)^{-a-1}e^{-ab}\gamma(a+1,-bu), \tag{2.4}$$

and 3.382.4:

$$\int_0^\infty (x+u)^a e^{-bx}\,dx = b^{-a-1}e^{bu}\Gamma(a+1,bu), \tag{2.5}$$

• 3.382.1
• 3.382.4

are evaluated by making linear changes of variables and reducing it to 3.381.1 or 3.381.3.

The change of variables $t = ce^{-v}$ yields

$$\int_y^\infty \exp(-ce^{-v}-av)\,dv = c^{-a}\gamma(a,x), \tag{2.6}$$

where $y = \ln(c/x)$. The special case $x = c$ produces

$$\int_0^\infty \exp(-ce^{-v}-av)\,dv = c^{-a}\gamma(a,c). \tag{2.7}$$

This appear as 3.331.1. • 3.331.1

Similarly, the change $t = ce^{v}$ yields

$$\int_y^\infty \exp(-ce^{v}+av)\,dv = c^{-a}\Gamma(a,x), \tag{2.8}$$

where $y = \ln(x/c)$. The special case $x = c$ produces

$$\int_0^\infty \exp(-ce^{v}+av)\,dv = c^{-a}\Gamma(a,c). \tag{2.9}$$

This appear as 3.331.2. • 3.331.2

3. An infinite series representation

The evaluation 3.381.2:

$$\begin{aligned}\int_0^x t^{a-1}e^{-t}\,dt &= \sum_{k=0}^\infty (-1)^k \frac{x^{a+k}}{k!\,(a+k)} \\ &= e^{-x}\sum_{k=0}^\infty \frac{x^{a+k}}{a(a+1)(a+2)\cdots(a+k)}.\end{aligned}$$

• 3.381.2

gives a series representation of the incomplete gamma function $\gamma(a,x)$. To obtain the first expression simply expand the exponential to get

$$\begin{aligned} I &= \sum_{k=0}^\infty \frac{(-1)^k}{k!}\int_0^x t^{a+k-1}\,dt \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{k!(a+k)}.\end{aligned}$$

To obtain the second expression for $\gamma(a,x)$, let $s = x-t$ in the integral to produce

$$I = e^{-x}\int_0^x (x-s)^{a-1}e^{s}, ds. \tag{3.1}$$

Now expand the exponentila and let $y = xs$ to obtain

$$\gamma(a,x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^{k+a}}{k!} \int_0^1 y^k (1-y)^{a-1} \, dy. \tag{3.2}$$

Now identify the integral as

$$B(k+1,a) = \frac{k! \, (a-1)!}{(a+p)!} \tag{3.3}$$

and the result is established.

Acknowledgments. The author acknowledges the partial support of NSF-DMS 0409968.

References

[1] I.S. Gradshteyn and I.M. Rhyzik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.

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