

# THE $p$ -ADIC VALUATION OF $k$ -CENTRAL BINOMIAL COEFFICIENTS

ARMIN STRAUB, VICTOR H. MOLL, AND TEWODROS AMDEBERHAN

ABSTRACT. The coefficients  $c(n, k)$  defined by

$$(1 - k^2 x)^{-1/k} = \sum_{n \geq 0} c(n, k) x^n$$

reduce to the central binomial coefficients  $\binom{2n}{n}$  for  $k = 2$ . Motivated by a question of H. Montgomery and H. Shapiro for the case  $k = 3$ , we prove that  $c(n, k)$  are integers and study their divisibility properties.

## 1. INTRODUCTION

In a recent issue of this journal, Hugh Montgomery and Harold S. Shapiro proposed the following problem (Problem 11380, August-September 2008):

For  $x \in \mathbb{R}$ , let

$$(1.1) \quad \binom{x}{n} = \frac{1}{n!} \prod_{j=0}^{n-1} (x - j).$$

For  $n \geq 1$ , let  $a_n$  be the numerator and  $q_n$  the denominator of the rational number  $\binom{-1/3}{n}$  expressed as a reduced fraction, with  $q_n > 0$ .

- (1) Show that  $q_n$  is a power of 3.
- (2) Show that  $a_n$  is odd if and only if  $n$  is a sum of distinct powers of 4.

Our approach to this problem employs Legendre's remarkable expression [7]:

$$(1.2) \quad \nu_p(n!) = \frac{n - s_p(n)}{p - 1},$$

that relates the  $p$ -adic valuation of factorials to the sum of digits of  $n$  in base  $p$ . For  $m \in \mathbb{N}$  and a prime  $p$ , the  $p$ -adic valuation of  $m$ , denoted by  $\nu_p(m)$ , is the highest power of  $p$  that divides  $m$ . The expansion of  $m \in \mathbb{N}$  in base  $p$  is written as

$$(1.3) \quad m = a_0 + a_1 p + \cdots + a_d p^d,$$

---

*Date:* September 26, 2008.

*1991 Mathematics Subject Classification.* Primary 11A51, Secondary 11A63.

*Key words and phrases.* Central binomial, generating functions, valuations.

with integers  $0 \leq a_j \leq p-1$  and  $a_d \neq 0$ . The function  $s_p$  in (1.2) is defined by

$$(1.4) \quad s_p(m) := a_0 + a_1 + \cdots + a_d.$$

Since, for  $n > 1$ ,  $\nu_p(n) = \nu_p(n!) - \nu_p((n-1)!)$ , it follows from (1.2) that

$$(1.5) \quad \nu_p(n) = \frac{1 + s_p(n-1) - s_p(n)}{p-1}.$$

The  $p$ -adic valuations of binomial coefficients can be expressed in terms of the function  $s_p$ :

$$(1.6) \quad \nu_p \left( \binom{n}{k} \right) = \frac{s_p(k) + s_p(n-k) - s_p(n)}{p-1}.$$

In particular, for the central binomial coefficients  $C_n := \binom{2n}{n}$  and  $p = 2$ , we have

$$(1.7) \quad \nu_2(C_n) = 2s_2(n) - s_2(2n) = s_2(n).$$

Therefore,  $C_n$  is always even and  $\frac{1}{2}C_n$  is odd precisely whenever  $n$  is a power of 2. This is a well-known result.

The central binomial coefficients  $C_n$  have the generating function

$$(1.8) \quad (1-4x)^{-1/2} = \sum_{n \geq 0} C_n x^n.$$

The binomial theorem shows that the numbers in the Montgomery-Shapiro problem bear a similar generating function

$$(1.9) \quad (1-9x)^{-1/3} = \sum_{n \geq 0} \binom{-\frac{1}{3}}{n} (-9x)^n.$$

It is natural to consider the coefficients  $c(n, k)$  defined by

$$(1.10) \quad (1-k^2x)^{-1/k} = \sum_{n \geq 0} c(n, k) x^n,$$

which include the central binomial coefficients as a special case. We call  $c(n, k)$  the  $k$ -central binomial coefficients. The expression

$$(1.11) \quad c(n, k) = (-1)^n \binom{-\frac{1}{k}}{n} k^{2n}$$

comes directly from the binomial theorem. Thus, the Montgomery-Shapiro question from (1.1) deals with arithmetic properties of

$$(1.12) \quad \binom{-\frac{1}{3}}{n} = (-1)^n \frac{c(n, 3)}{3^{2n}}.$$

2. THE INTEGRALITY OF  $c(n, k)$

It is a simple matter to verify that the coefficients  $c(n, k)$  are rational numbers. The expression produced in the next proposition is then employed to prove that  $c(n, k)$  are actually integers. The next section will explore divisibility properties of the integers  $c(n, k)$ .

**Proposition 2.1.** *The coefficient  $c(n, k)$  is given by*

$$(2.1) \quad c(n, k) = \frac{k^n}{n!} \prod_{m=1}^{n-1} (1 + km).$$

*Proof.* The binomial theorem yields

$$\begin{aligned} (1 - k^2x)^{-1/k} &= \sum_{n \geq 0} \binom{-\frac{1}{k}}{n} (-k^2x)^n \\ &= \sum_{n \geq 0} \frac{k^n}{n!} \left( \prod_{m=1}^{n-1} (1 + km) \right) x^n, \end{aligned}$$

and (2.1) has been established.  $\square$

An alternative proof of the previous result is obtained from the simple recurrence

$$(2.2) \quad c(n + 1, k) = \frac{k(1 + kn)}{n + 1} c(n, k), \quad \text{for } n \geq 0,$$

and its initial condition  $c(0, k) = 1$ . To prove (2.2), simply differentiate (1.10) to produce

$$(2.3) \quad k(1 - k^2x)^{-1/k-1} = \sum_{n \geq 0} (n + 1)c(n + 1, k)x^n$$

and multiply both sides by  $1 - k^2x$  to get the result.

**Note.** The coefficients  $c(n, k)$  can be written in terms of the Beta function as

$$(2.4) \quad c(n, k) = \frac{k^{2n}}{nB(n, 1/k)}.$$

This expression follows directly by writing the product in (2.1) in terms of the Pochhammer symbol  $(a)_n = a(a + 1) \cdots (a + n - 1)$  and the identity

$$(2.5) \quad (a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}.$$

The proof employs only the most elementary properties of the Euler's Gamma and Beta functions. The reader can find details in [1]. The conclusion is that we have an integral expression for  $c(n, k)$ , given by

$$(2.6) \quad c(n, k) \int_0^1 (1 - u^{1/n})^{1/k-1} du = k^{2n}.$$

It is unclear how to use it to further investigate  $c(n, k)$ .

In the case  $k = 2$ , we have that  $c(n, 2) = C_n$  is a positive integer. This result extends to all values of  $k$ .

**Theorem 2.2.** *The coefficient  $c(n, k)$  is a positive integer.*

*Proof.* First observe that if  $p$  is a prime dividing  $k$ , then the product in (1.10) is relatively prime to  $p$ . Therefore we need to check that  $\nu_p(n!) \leq \nu_p(k^n)$ . This is simple:

$$(2.7) \quad \nu_p(n!) = \frac{n - s_p(n)}{p-1} \leq n \leq \nu_p(k^n).$$

Now let  $p$  be a prime not dividing  $k$ . Clearly,

$$(2.8) \quad \nu_p(c(n, k)) = \nu_p\left(\prod_{m < n} (1 + km)\right) - \nu_p\left(\prod_{m < n} (1 + m)\right).$$

To prove that  $c(n, k)$  is an integer, we compare the  $p$ -adic valuations of  $1 + km$  and  $1 + m$ . Observe that  $1 + m$  is divisible by  $p^\alpha$  if and only if  $m$  is of the form  $\lambda p^\alpha - 1$ . On the other hand,  $1 + km$  is divisible by  $p^\alpha$  precisely when  $m$  is of the form  $\lambda p^\alpha - i_{p^\alpha}(k)$ , where  $i_{p^\alpha}(k)$  denotes the inverse of  $k$  modulo  $p^\alpha$  in the range  $1, 2, \dots, p^\alpha - 1$ . Thus,

$$(2.9) \quad \nu_p(c(n, k)) = \sum_{\alpha \geq 1} \left\lfloor \frac{n + i_{p^\alpha}(k) - 1}{p^\alpha} \right\rfloor - \left\lfloor \frac{n}{p^\alpha} \right\rfloor.$$

The claim now follows from  $i_{p^\alpha}(k) \geq 1$ .  $\square$

Next, Theorem 2.2 will be slightly strengthened and an alternative proof be provided.

**Theorem 2.3.** *For  $n > 0$ , the coefficient  $c(n, k)$  is a positive integer divisible by  $k$ .*

*Proof.* Expanding the right hand side of the identity

$$(2.10) \quad (1 - k^2x)^{-1} = \left((1 - k^2x)^{-1/k}\right)^k$$

by the Cauchy product formula gives

$$(2.11) \quad \sum_{i_1 + \dots + i_k = m} c(i_1, k) c(i_2, k) \cdots c(i_k, k) = k^{2m},$$

where the multisum runs through all the  $k$ -tuples of non-negative integers. Obviously  $c(0, k) = 1$  and it is easy to check that  $c(1, k) = k$ . We proceed by induction on  $n$ , so we assume the assertion is valid for  $c(1, k), c(2, k), \dots, c(n-1, k)$ . We prove the same is true for  $c(n, k)$ . To this end, break up (2.11) as

$$(2.12) \quad kc(n, k) + \sum_{\substack{i_1 + \dots + i_k = n \\ 0 \leq i_j < n}} c(i_1, k) c(i_2, k) \cdots c(i_k, k) = k^{2n}.$$

Hence by the induction assumption  $kc(n, k)$  is an integer.

To complete the proof, divide (2.12) through by  $k^2$  and rewrite as follows

$$(2.13) \quad \frac{c(n, k)}{k} = k^{2n-2} - \frac{1}{k^2} \sum_{\substack{i_1+\dots+i_k=n \\ 0 \leq i_j < n}} c(i_1, k)c(i_2, k) \cdots c(i_k, k).$$

The key point is that each summand in (2.13) contains *at least two* terms, each one divisible by  $k$ . □

**Note.** W. Lang [6] has studied the numbers appearing in the generating function

$$(2.14) \quad c2(l; x) := \frac{1 - (1 - l^2x)^{1/l}}{lx},$$

that bears close relation to the case  $k = -l < 0$  of equation (1.10). The special case  $l = 2$  yields the Catalan numbers. The author establishes the integrality of the coefficients in the expansion of  $c2$  and other related functions.

### 3. THE VALUATION OF $c(n, k)$

We consider now the  $p$ -adic valuation of  $c(n, k)$ . The special case when  $p$  divides  $k$  is easy, so we deal with it first.

**Proposition 3.1.** *Let  $p$  be a prime that divides  $k$ . Then*

$$(3.1) \quad \nu_p(c(n, p)) = \nu_p(k)n - \frac{n - s_p(n)}{p - 1}.$$

*Proof.* The  $p$ -adic valuation of  $c(n, p)$  is given by

$$(3.2) \quad \nu_p(c(n, p)) = \nu_p(k)n - \nu_p(n!) = \nu_p(k)n - \frac{n - s_p(n)}{p - 1}.$$

Finally note that  $s_p(n) = O(\log n)$ . □

**Note.** For  $p, k \neq 2$ , we have  $\nu_p(c(n, p)) \sim \left(\nu_p(k) - \frac{1}{p-1}\right)n$ , as  $n \rightarrow \infty$ .

We now turn attention to the case where  $p$  does not divide  $k$ . Under this assumption, the congruence  $kx \equiv 1 \pmod{p^\alpha}$  has a solution. Elementary arguments of  $p$ -adic analysis can be used to produce a  $p$ -adic integer that yields the inverse of  $k$ . This construction proceeds as follows: first choose  $b_0$  in the range  $\{1, 2, \dots, p-1\}$  to satisfy  $kb_0 \equiv 1 \pmod{p}$ . Next, choose  $c_1$ , satisfying  $kc_1 \equiv 1 \pmod{p^2}$  and write it as  $c_1 = b_0 + kb_1$  with  $0 \leq b_1 \leq p-1$ . Proceeding in this manner, we obtain a sequence of integers  $\{b_j : j \geq 0\}$ , such that  $0 \leq b_j \leq p-1$  and the partial sums of the *formal object*  $x = b_0 + b_1p + b_2p^2 + \dots$  satisfy

$$(3.3) \quad k(b_0 + b_1p + \dots + b_{j-1}p^{j-1}) \equiv 1 \pmod{p^j}.$$

This is the standard definition of a  $p$ -adic integer and

$$(3.4) \quad i_{p^\infty}(k) = \sum_{j=0}^{\infty} b_j p^j$$

is the inverse of  $k$  in the ring of  $p$ -adic integers. The reader will find in [3] and [8] information about this topic.

**Note.** It is convenient to modify the notation in (3.4) and write it as

$$(3.5) \quad i_{p^\infty}(k) = 1 + \sum_{j=0}^{\infty} b_j p^j$$

which is always possible since the first coefficient cannot be zero. The reader is invited to check that, when doing so, the  $b_j$  are periodic in  $j$  with period the multiplicative order of  $p$  in  $\mathbb{Z}/k\mathbb{Z}$ . Furthermore, the  $b_j$  take values amongst  $\lfloor p/k \rfloor, \lfloor 2p/k \rfloor, \dots, \lfloor (k-1)p/k \rfloor$ . This will be exemplified in the case  $k = 3$  later.

The analysis of  $\nu_p(c(n, k))$  for those primes  $p$  not dividing  $k$  begins with a characterization of those indices for which  $\nu_p(c(n, k)) = 0$ , that is,  $p$  does not divide  $c(n, k)$ . The result is expressed in terms of the expansions of  $n$  in base  $p$ , written as

$$(3.6) \quad n = a_0 + a_1 p + a_2 p^2 + \dots + a_d p^d,$$

and the  $p$ -adic expansion of the inverse of  $k$  as given by (3.5).

**Theorem 3.2.** *Let  $p$  be a prime that does not divide  $k$ . Then  $\nu_p(c(n, k)) = 0$  if and only if  $a_j + b_j < p$  for all  $j$  in the range  $1 \leq j \leq d$ .*

*Proof.* It follows from (2.9) that  $c(n, k)$  is not divisible by  $p$  precisely when

$$(3.7) \quad \left\lfloor \frac{1}{p^\alpha} \left( n + \sum_j b_j p^j \right) \right\rfloor = \left\lfloor \frac{n}{p^\alpha} \right\rfloor,$$

for all  $\alpha \geq 1$ , or equivalently, if and only if

$$(3.8) \quad \sum_{j=0}^{\alpha-1} (a_j + b_j) p^j < p^\alpha,$$

for all  $\alpha \geq 1$ . An inductive argument shows that this is equivalent to the condition  $a_j + b_j < p$  for all  $j$ . Naturally, the  $a_j$  vanish for  $j > d$ , so it is sufficient to check  $a_j + b_j < p$  for all  $j \leq d$ .  $\square$

**Corollary 3.3.** *For all primes  $p > k$  and  $d \in \mathbb{N}$ , we have  $\nu_p(c(p^d, k)) = 0$ .*

*Proof.* The coefficients of  $n = p^d$  in Theorem 3.2 are  $a_j = 0$  for  $0 \leq j \leq d-1$  and  $a_d = 1$ . Therefore the restrictions on the coefficients  $b_j$  become  $b_j < p$

for  $0 \leq j \leq d - 1$  and  $b_d < p - 1$ . It turns out that  $b_j \neq p - 1$  for all  $j \in \mathbb{N}$ . Otherwise, for some  $r \in \mathbb{N}$ , we have  $b_r = p - 1$  and the equation

$$(3.9) \quad k \left( 1 + \sum_{j=0}^{r-1} b_j p^j + b_r p^r \right) \equiv k \left( 1 + \sum_{j=0}^{r-1} b_j p^j - p^r \right) \equiv 1 \pmod{p^{r+1}},$$

is impossible in view of

$$(3.10) \quad -kp^r < k \left( 1 + \sum_{j=0}^{r-1} b_j p^j - p^r \right) < 0.$$

□

Now we return again to the Montgomery-Shapiro question. The identity (1.12) shows that the denominator  $q_n$  is a power of 3. We now consider the indices  $n$  for which  $c(n, 3)$  is odd and provide a proof of the second part of their problem.

**Corollary 3.4.** *The coefficient  $c(n, 3)$  is odd precisely when  $n$  is a sum of distinct powers of 4.*

*Proof.* The result follows from Theorem 3.2 and the explicit formula

$$(3.11) \quad i_{2^\infty}(3) = 1 + \sum_{j=0}^{\infty} 2^{2j+1},$$

for the inverse of 3, so that  $b_{2j} = 0$  and  $b_{2j+1} = 1$ . Therefore, if  $c(n, 3)$  is odd, the theorem now shows that  $a_j = 0$  for  $j$  odd, as claimed. □

More generally, the discussion of  $\nu_p(c(n, 3)) = 0$  is divided according to the residue of  $p$  modulo 3. This division is a consequence of the fact that for  $p = 3u + 1$ , we have

$$(3.12) \quad i_{p^\infty}(3) = 1 + 2u \sum_{m=0}^{\infty} p^m,$$

and for  $p = 3u + 2$ , one computes  $p^2 = 3(3u^2 + 4u + 1) + 1$ , to conclude that

$$(3.13) \quad i_{p^\infty}(3) = 1 + 2(3u^2 + 4u + 1) \sum_{m=0}^{\infty} p^{2m} = 1 + \sum_{m=0}^{\infty} up^{2m} + (2u + 1)p^{2m+1}.$$

**Theorem 3.5.** *Let  $p \neq 3$  be a prime and  $n = a_0 + a_1p + a_2p^2 + \dots + a_dp^d$  as before. Then  $p$  does not divide  $c(n, 3)$  if and only if the  $p$ -adic digits of  $n$  satisfy*

$$(3.14) \quad a_j < \begin{cases} p/3 & \text{if } j \text{ is odd or } p = 3u + 1, \\ 2p/3 & \text{otherwise.} \end{cases}$$

For general  $k$  we have the following analogous statement.

**Theorem 3.6.** *Let  $p = ku + 1$  be a prime. Then  $p$  does not divide  $c(n, k)$  if and only if the  $p$ -adic digits of  $n$  are less than  $p/k$ .*

Observe that Theorem 3.6 implies the following well-known property of the central binomial coefficients:  $C_n$  is not divisible by  $p \neq 2$  if and only if the  $p$ -adic digits of  $n$  are less than  $p/2$ .

Now we return to (2.9) which will be written as

$$(3.15) \quad \nu_p(c(n, k)) = \sum_{\alpha \geq 0} \left\lfloor \frac{1}{p^{\alpha+1}} \sum_{m=0}^{\alpha} (a_m + b_m)p^m \right\rfloor.$$

From here, we bound

$$(3.16) \quad \sum_{m=0}^{\alpha} (a_m + b_m)p^m \leq \sum_{m=0}^{\alpha} (2p - 2)p^m = 2(p^{\alpha+1} - 1) < 2p^{\alpha+1}.$$

Therefore, each summand in (3.15) is either 0 or 1. The  $p$ -adic valuation of  $c(n, p)$  counts the number of 1's in this sum. This proves the final result.

**Theorem 3.7.** *Let  $p$  be a prime that does not divide  $k$ . Then, with the previous notation for  $a_m$  and  $b_m$ , we have that  $\nu_p(c(n, k))$  is the number of indices  $m$  such that either*

- $a_m + b_m \geq p$  or
- there is  $j \leq m$  such that  $a_{m-i} + b_{m-i} = p - 1$  for  $0 \leq i \leq j - 1$  and  $a_{m-j} + b_{m-j} \geq p$ .

**Corollary 3.8.** *Let  $p$  be a prime that does not divide  $k$ , and write  $n = \sum a_m p^m$  and  $i_{p^\infty}(k) = 1 + \sum b_m p^m$ , as before. Let  $v_1$  and  $v_2$  be the number of indices  $m$  such that  $a_m + b_m \geq p$  and  $a_m + b_m \geq p - 1$ , respectively. Then*

$$(3.17) \quad v_1 \leq \nu_p(c(n, k)) \leq v_2.$$

#### 4. A $q$ -GENERALIZATION OF $c(n, k)$

A standard procedure to generalize an integer expression is to replace  $n \in \mathbb{N}$  by the polynomial

$$(4.1) \quad [q]_n := \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \dots + q^{n-1}.$$

The original expression is recovered as the limiting case  $q \rightarrow 1$ . For example, the factorial  $n!$  is extended to the polynomial

$$(4.2) \quad [n]_q! := [n]_q [n-1]_q \dots [2]_q [1]_q = \prod_{j=1}^n \frac{1 - q^j}{1 - q}.$$

The reader will find in [5] an introduction to this  $q$ -world.

In this spirit we generalize the integers

$$(4.3) \quad c(n, k) = \frac{k^n}{n!} \prod_{m=0}^{n-1} (km + 1) = \prod_{m=1}^n \frac{k(k(m-1) + 1)}{m},$$

into the  $q$ -world as

$$(4.4) \quad F_{n,k}(q) := \prod_{m=1}^n \frac{[km]_q [k(m-1) + 1]_q}{[m]_q^2}.$$

Note that this expression indeed gives  $c(n, k)$  as  $q \rightarrow 1$ . The corresponding extension of Theorem 2.2 is stated in the next result. The proof is similar to that given above, so it is left to the curious reader.

**Theorem 4.1.** *The function*

$$(4.5) \quad F_{n,k}(q) := \prod_{m=1}^n \frac{(1 - q^{km})(1 - q^{k(m-1)+1})}{(1 - q^m)^2}$$

*is a polynomial in  $q$  with integer coefficients.*

## 5. FUTURE DIRECTIONS

In this final section we discuss some questions related to the integers  $c(n, k)$ .

• **A combinatorial interpretation.** The integers  $c(n, 2)$  are given by the central binomial coefficients  $C_n = \binom{2n}{n}$ . These coefficients appear in many counting situations:  $C_n$  gives the number of walks of length  $2n$  on an infinite linear lattice that begin and end at the origin. Moreover, they provide the exact answer for the elementary sum

$$(5.1) \quad \sum_{k=0}^n \binom{n}{k}^2 = C_n.$$

Is it possible to produce similar results for  $c(n, k)$ , with  $k \neq 2$ ? In particular, what do the numbers  $c(n, k)$  count?

• **A further generalization.** The polynomial  $F_{n,k}(q)$  can be written as

$$(5.2) \quad F_{n,k}(q) = \frac{(1 - q)}{(1 - q^{kn+1})} \prod_{m=1}^n \frac{(1 - q^{km})(1 - q^{km+1})}{(1 - q^m)^2}$$

which suggests the extension

$$(5.3) \quad G_{n,k}(q, t) := \frac{(1 - q)}{(1 - tq^{kn})} \prod_{m=1}^n \frac{(1 - q^{km})(1 - tq^{km})}{(1 - q^m)^2}$$

so that  $F_{n,k}(q) = G_{n,k}(q, q)$ . Observe that  $G_{n,k}(q, t)$  is not always a polynomial. For example,

$$(5.4) \quad G_{2,1}(q, t) = \frac{1 - qt}{1 - q^2}.$$

On the other hand,

$$(5.5) \quad G_{1,2}(q, t) = q + 1.$$

The following functional equation is easy to establish.

**Proposition 5.1.** *The function  $G_{n,k}(q, t)$  satisfies*

$$(5.6) \quad G_{n,k}(q, tq^k) = \frac{(1 - q^{kn}t)}{(1 - q^kt)} G_{n,k}(q, t).$$

The reader is invited to explore its properties. In particular, find minimal conditions on  $n$  and  $k$  to guarantee that  $G_{n,k}(q, t)$  is a polynomial.

Consider now the function

$$(5.7) \quad H_{n,k,j}(q) := G_{n,k}(q, q^j)$$

that extends  $F_{n,k}(q) = H_{n,k,1}(q)$ . The following statement predicts the situation where  $H_{n,k,j}(q)$  is a polynomial.

**Problem.** Show that  $H_{n,k,j}(q)$  is a polynomial precisely if the indices satisfy  $k \equiv 0 \pmod{\gcd(n, j)}$ .

• **A result of Erdős, Graham, Ruzsa and Straus.** In this paper we have explored the conditions on  $n$  that result in  $\nu_p(c(n, k)) = 0$ . Given two distinct primes  $p$  and  $q$ , P. Erdős et al. [2] discuss the existence of indices  $n$  for which  $\nu_p(C_n) = \nu_q(C_n) = 0$ . Recall that by Theorem 3.6 such numbers  $n$  are characterized by having  $p$ -adic digits less than  $p/2$  and  $q$ -adic digits less than  $q/2$ . The following result of [2] proves the existence of infinitely many such  $n$ .

**Theorem 5.2.** *Let  $A, B \in \mathbb{N}$  such that  $A/(p-1) + B/(q-1) \geq 1$ . Then there exist infinitely many numbers  $n$  with  $p$ -adic digits  $\leq A$  and  $q$ -adic digits  $\leq B$ .*

This leaves open the question for  $k > 2$  whether or not there exist infinitely many numbers  $n$  such that  $c(n, k)$  is neither divisible by  $p$  nor by  $q$ . The extension to more than two primes is open even in the case  $k = 2$ . In particular, a prize of \$1000 has been offered by R. Graham for just showing that there are infinitely many  $n$  such that  $C_n$  is coprime to  $105 = 3 \cdot 5 \cdot 7$ . On the other hand, it is conjectured that there are only finitely many indices  $n$  such that  $C_n$  is not divisible by any of 3, 5, 7 and 11.

Finally, we remark that Erdős et al. conjectured in [2] that the central binomial coefficients  $C_n$  are never squarefree for  $n > 4$  which has been proved

by Granville and Ramare in [4]. Define

$$(5.8) \quad \tilde{c}(n, k) := \text{Numerator} \left( k^{-n} c(n, k) \right).$$

We have *some* empirical evidence which suggests the existence of an index  $n_0(k)$ , such that  $\tilde{c}(n, k)$  is not squarefree for  $n \geq n_0(k)$ . The value of  $n_0(k)$  could be large. For instance

$$\begin{aligned} \tilde{c}(178, 5) = & 10233168474238806048538224953529562250076040177895261 \\ & 58561031939088200683714293748693318575050979745244814 \\ & 765545543340634517536617935393944411414694781142 \end{aligned}$$

is squarefree, so that  $n_0(5) \geq 178$ . The numbers  $\tilde{c}(n, k)$  present new challenges, even in the case  $k = 2$ . Recall that  $\frac{1}{2}C_n$  is odd if and only if  $n$  is a power of 2. Therefore,  $C_{786}$  is not squarefree. On the other hand, the complete factorization of  $C_{786}$  shows that  $\tilde{c}(786, 2)$  is squarefree. We conclude that  $n_0(2) \geq 786$ .

**Acknowledgments.** The work of the second author was partially funded by NSF-DMS 0409968. The first author was partially supported, as a graduate student, by the same grant.

#### REFERENCES

- [1] G. Boros and V. Moll. *Irresistible Integrals*. Cambridge University Press, New York, 1st edition, 2004.
- [2] P. Erdős, R. Graham, I. Ruzsa, and E. Straus. On the prime factors of  $\binom{2n}{n}$ . *Math. Comp.*, 29:83–92, 1975.
- [3] F. Gouvea. *p-adic numbers*. Springer Verlag, 2nd edition, 1997.
- [4] A. Granville and O. Ramare. Explicit bounds on exponential sums and the scarcity of squarefree binomial coefficients. *Mathematika*, 43:73–107, 1996.
- [5] V. Kac and P. Cheung. *Quantum Calculus*. Springer-Verlag Universitytext, New York, 2002.
- [6] W. Lang. On generalizations of the Stirling numbers. *Journal of Integer Sequences*, 3:Article 00.2.4; 18 pages, 2000.
- [7] A. M. Legendre. *Theorie des Nombres*. Firmin Didot Freres, Paris, 1830.
- [8] M. Ram Murty. *Introduction to p-adic Analytic Number Theory*, volume 27 of *Studies in Advanced Mathematics*. American Mathematical Society, 1st edition, 2002.

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118  
*E-mail address:* `astraub@math.tulane.edu`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118  
*E-mail address:* `vhm@math.tulane.edu`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118  
*E-mail address:* `tamdeberhan@math.tulane.edu`