THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 24: EVALUATION USING THE LERCH ZETA FUNCTION. PRELIMINARY VERSION: LAST UPDATE SEPTEMBER 24, 2006.

VICTOR H. MOLL

ABSTRACT. The table of Gradshteyn and Rhyzik contains many integrals that can be evaluated using the Lerch zeta function. Some examples are discussed.

1. Introduction

The Lerch zeta function is defined in [1], entry 9.950, by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^s}. \tag{1.1}$$

The series converges for $|z| < 1$.

2. A first integral representation

• 3.411.6 The table [1] contains as 3.411.6 the representation

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}\, e^{-xv}\, dx}{1-ze^{-x}}. \tag{2.1}$$

To establish this representation, expand the integrand as

$$\frac{x^{s-1}\, e^{-xv}}{1-ze^{-x}} = x^{s-1} e^{-xv} \sum_{n=0}^{\infty} z^n e^{-nx}, \tag{2.2}$$

and then integrate term by term.

The evaluation 3.411.22:

$$\int_0^{\infty} \frac{x^{p-1}\, dx}{e^{rx}-q} = \frac{\Gamma(p)}{r^p} \Phi(q,p,1) \tag{2.3}$$

• 3.411.22 can be established by expanding the integrand. Observe first that r is a fake parameter: the change $t = rx$ shows that this evaluation is equivalent to the one for $r = 1$:

$$\int_0^{\infty} \frac{t^{p-1}\, dt}{e^t - q} = \Gamma(p)\Phi(q,p,1). \tag{2.4}$$

Date: September 25, 2006.
1991 *Mathematics Subject Classification.* Primary 33.
Key words and phrases. Integrals.

Expanding the integrand as

$$\frac{t^{p-1}}{e^t - q} = \frac{t^{p-1}e^{-t}}{1 - qe^{-t}} = x^{p-1}e^{-t}\sum_{j=0}^{\infty} q^j e^{-jt},$$

we obtain

$$\begin{aligned} \int_0^\infty \frac{x^{p-1}\,dx}{e^t - q} &= \sum_{j=0}^{\infty} q^j \int_0^\infty t^{p-1}e^{-(j+1)t}\,dt \\ &= \sum_{j=0}^{\infty} \frac{q^j\,\Gamma(p)}{(j+1)^p} \\ &= \Gamma(p)\Phi(q,p,1). \end{aligned}$$

Acknowledgments. The author acknowledges the partial support of NSF-DMS 0409968.

References

[1] I.S. Gradshteyn and I.M. Rhyzik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@math.tulane.edu