THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 26: COMBINATIONS OF LOGARITHMS AND ALGEBRAIC FUNCTIONS. PRELIMINARY VERSION: LAST UPDATE SEPTEMBER 26, 2006.

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Abstract. We present a systematic derivation of some of the definite integrals in the classical table of Gradshteyn and Rhyzik where the integrand is a combination of logarithms and algebraic functions.

1. Introduction

The table of integrals [1] contains some evaluations where the integrand is a combination of logarithms and an algebraic function. For example 4.241.9:

$$\int_0^1 \sqrt{1-x^2}\,\ln x\,dx = -\frac{\pi}{8}(1+2\ln 2). \tag{1.1}$$

• 4.241.9 The goal of this paper is to present a systematic derivation of these evaluations.

2. A reduction to an exponential integral

The change of variables $x = e^{-u}$ produces

$$\int_0^1 \sqrt{1-x^2}\,\ln x\,dx = -\int_0^\infty ue^{-u}\sqrt{1-e^{-2u}}\,du. \tag{2.1}$$

• 3.451.2

This integral appears as entry 3.451.2 of [1] and the value

$$\int_0^\infty ue^{-u}\sqrt{1-e^{-2u}}\,du = \frac{\pi}{8}(1+2\ln 2) \tag{2.2}$$

was established in [2]. This proves (1.1).

Similarly, from 3.452.4:

$$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^{2x}-1}} = 1-\ln 2, \tag{2.3}$$

and the change of variables $t = e^x$ we obtain

$$\int_1^\infty \frac{\ln t\,dt}{t^2\,\sqrt{t^2-1}} = 1-\ln 2. \tag{2.4}$$

• 4.241.8

This appears as 4.241.8 in [1].

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3. A reduction to a combination of logarithm and trigonometry

Entry 4.241.7 states that

$$\int_0^1 \frac{\ln x\, dx}{\sqrt{1-x^2}} = -\frac{\pi}{2} \ln 2. \tag{3.1}$$

The change of variables $x = \sin t$ produces • 4.241.7

$$\int_0^1 \frac{\ln x\, dx}{\sqrt{1-x^2}} = \int_0^{\pi/2} \ln \sin t \, dt. \tag{3.2}$$

This integral is shown to be $-\frac{\pi}{2} \ln 2$ in [3].

Entry 4.243

$$\int_0^1 \frac{x \, \ln x}{\sqrt{1-x^4}} = -\frac{\pi}{8} \ln 2, \tag{3.3}$$

is reduced to (3.1) by the change of variables $t = x^2$. • 4.243

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References

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