THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 25: COMBINATIONS OF POWERS AND ALGEBRAIC FUNCTIONS OF EXPONENTIALS. PRELIMINARY VERSION: LAST UPDATE SEPTEMBER 27, 2006.

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Abstract. The table of Gradshteyn and Rhyzik contains many integrals where the integrand is a power times and algebraic function of the exponential. Some examples are discussed.

1. Introduction

The table [1] contains many evaluations of definite integrals of the form

$$I = \int_0^\infty x^n A(e^{-x})\, dx \tag{1.1}$$

where A is an algebraic function. These are functions that satisfy a polynomial equation $P(x, A(x)) = 0$.

2. A first family of examples

The first evaluation described here is 3.451.1:

$$\int_0^\infty x e^{-x} \sqrt{1-e^{-x}}\, dx = \frac{4}{9}(4 - 3\ln 2). \tag{2.1}$$

• 3.451.1 This is the case $a = 1$ of the integral

$$I(a) = \int_0^\infty x e^{-x} \sqrt{1-e^{-ax}}\, dx. \tag{2.2}$$

In 3.451.2 we find the evaluation of $I(2)$:

$$\int_0^\infty x e^{-x} \sqrt{1-e^{-2x}}\, dx = \frac{\pi}{8}(1 + 2\ln 2). \tag{2.3}$$

• 3.451.2 We first provide an answer in terms of an infinite series. For special values of a we then provide an elementary method to sum the series.

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The binomial theorem shows that

$$\begin{aligned} \sqrt{1-e^{-ax}} &= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-1)^k e^{-akx} \\ &= -\sum_{k=0}^{\infty} \frac{(2k)!}{(2k-1)\, 2^{2k}\, k!^2} e^{-akx}. \end{aligned}$$

Therefore we have

$$\begin{aligned} I(a) &= -\sum_{k=0}^{\infty} \frac{(2k)!}{(2k-1)2^{2k}k!^2} \int_0^{\infty} x e^{-(1+ak)x}\, dx \\ &= -\sum_{k=0}^{\infty} \frac{(2k)!}{(2k-1)2^{2k}k!^2} \frac{1}{(1+ak)^2} \end{aligned}$$

that can be written as

$$I(a) = 1 - \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k+1}(k+1)!k!\,(b+ak)^2}, \tag{2.4}$$

with $b = 1+a$.

The first evaluation requires $I(1) = 1 - S_1$, where

$$S_1 = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k+1}(k+2)^2(k+1)k!^2}. \tag{2.5}$$

To evaluate this um by elementary means we start with

$$f(x) = \sum_{k=0}^{\infty} \frac{(2k)!}{k!^2} x^k = \frac{1}{\sqrt{1-4x}}, \tag{2.6}$$

where the last evaluation comes from the binomial theorem.

Define

$$g(x) = \int_0^x f(t)\, dt = \sum_{k=0}^{\infty} \frac{(2k)!}{k!^2} \frac{x^{k+1}}{k+1}. \tag{2.7}$$

An elementary calculation shows that

$$g(x) = \frac{1}{2}(1 - \sqrt{1-4x}). \tag{2.8}$$

Now define

$$h(x) = \int_0^x g(t)\, dt = \sum_{k=0}^{\infty} \frac{(2k)!}{k!^2} \frac{x^{k+2}}{(k+1)(k+2)}, \tag{2.9}$$

and also

$$h(x) = \frac{x}{2} - \frac{1}{12} + \frac{1}{12}(1-4x)^{3/2}. \tag{2.10}$$

Finally define

$$w(x) = \int_0^x \frac{h(t)}{t}\, dt = \sum_{k=0}^{\infty} \frac{(2k)!}{k!^2} \frac{x^{k+2}}{(k+1)(k+2)^2}, \tag{2.11}$$

and comparing this to the series defining S_1 in (2.5) we obtain

$$S_1 = 8w(1/4). \tag{2.12}$$

Now observe that

$$w(x) = \int_0^x \frac{h(t)}{t}\,dt = \frac{x}{2} + \frac{1}{12}J(x), \tag{2.13}$$

where

$$J(x) = \int_0^x \frac{(1-4t)^{3/2}-1}{t}\,dt. \tag{2.14}$$

The change of variables $u = \sqrt{1-4t}$ gives

$$J(x) = -2\int_{\sqrt{1-4x}}^1 \frac{u(1+u+u^2}{1+u}\,du, \tag{2.15}$$

and the further change of variables $v = 1+u$ gives

$$J(x) = -2\int_\sigma^2 (v^2 - 2v + 2 - 1/v)\,dv, \tag{2.16}$$

where $\sigma = 1 + \sqrt{1-4x}$.

This can be evaluated in elementary terms to produce

$$w(x) = \frac{1}{18}\left(-4 + 4\sqrt{1-4x} + x(9 - 4\sqrt{1-4x}) + 3\ln 2 - 3\ln(1+\sqrt{1-4x})\right).$$

In particular $w(1/4) = \frac{1}{18}(3\ln 2 - 7/4)$ and then

$$I(1) = 1 - S_1 = 1 - 8t(\tfrac{1}{4}) = \frac{4}{9}(4 - 3\ln 2) \tag{2.17}$$

as required.

We now use the same techniques to evaluate $I(2) = 1 - S_2$, where

$$S_2 = \sum_{k=0}^\infty \frac{(2k)!}{2^{2k+1}\,(k+1)!\,k!\,(2k+3)^2}. \tag{2.18}$$

We begin with

$$f(x) = \sum_{k=0}^\infty \frac{(2k)!}{k!^2}x^k = \frac{1}{\sqrt{1-4x}}, \tag{2.19}$$

and then evaluate

$$g(x) := \int_0^x f(t)dt = \sum_{k=0}^\infty \frac{(2k)!}{k!\,(k+1)!}x^{k+1}, \tag{2.20}$$

and as before

$$g(x) = \frac{1-\sqrt{1-4x}}{2}. \tag{2.21}$$

The next step is to form

$$h(x) = \int_0^x g(t^2)\,dt = \sum_{k=0}^\infty \frac{(2k)!}{(k+1)!\,k!}\,\frac{x^{2k+3}}{2k+3}. \tag{2.22}$$

Now

$$\begin{aligned} h(x) &= \frac{1}{2}\int_0^x (1-\sqrt{1-4t^2})\,dt \\ &= \frac{x}{2} - \frac{1}{2}\int_0^x \sqrt{1-4t^2}\,dt. \end{aligned}$$

Elementary changes of variables yield

$$h(x) = \frac{x}{2} - \frac{1}{8}\text{Arcsin}(2x) - \frac{x}{4}x\sqrt{1-4x^2}. \tag{2.23}$$

Define

$$w(x) = \int_0^x \frac{h(t)}{t}\,dt = \sum_{k=0}^{\infty} \frac{(2k)!}{(k+1)!\,k!}\frac{x^{2k+3}}{(2k+3)^2}, \tag{2.24}$$

so that $S_2 = 4w(1/2)$. Now,

$$\begin{aligned} w(x) &= \int_0^x \left(\frac{1}{2} - \frac{1}{8}\frac{\text{Arcsin}(2t)}{t} - \frac{1}{4}\sqrt{1-4t^2} \right) dt \\ &= \frac{x}{2} - \frac{1}{8}x\sqrt{1-4x^2} - \frac{1}{16}\text{Arcsin}(2x) - \frac{1}{8}\int_0^x \frac{\text{Arcsin}(2t)}{t}\,dt \end{aligned}$$

The change of variables $\varphi = \text{Arcsin}(2t)$ yields

$$\int_0^x \frac{\text{Arcsin}(2t)}{t}\,dt = \int_0^{\text{Arcsin}(2x)} \varphi\,\cot\varphi\,d\varphi. \tag{2.25}$$

We conclude that

$$w(\tfrac{1}{2}) = \frac{1}{4} - \frac{\pi}{12} - \frac{1}{8}\int_0^{\pi/2} \frac{\varphi\,d\varphi}{\tan\varphi}. \tag{2.26}$$

The evaluation

$$\int_0^{\pi/2} \frac{\varphi\,d\varphi}{\tan\varphi} = \frac{\pi}{2}\ln 2, \tag{2.27}$$

is presented in [2]. We conclude that

$$I(2) = \frac{\pi}{8}(1 + 2\ln 2), \tag{2.28}$$

as claimed.

3. A logarithmic scale

The integral

$$I(a) = \int_0^\infty xe^{-x}\sqrt{1-e^{-ax}}\,dx \tag{3.1}$$

can be expressed in many different forms. The change of variables $t = e^{-ax}$ yields

$$I(a) = -\frac{1}{a^2}\int_0^1 t^{1/a-1}\sqrt{1-t}\,\ln t\,dt, \tag{3.2}$$

so the value

$$I(1) = \frac{4}{9}(4 - 3\ln 2) \tag{3.3}$$

produces

$$\int_0^1 t^{1/a-1}\sqrt{1-t}\,\ln t\,dt = -\frac{4a^2}{9}(4-3\ln 2). \tag{3.4}$$

For instance, $a=3$ yields

$$\int_0^1 t^{-2/3}\sqrt{1-t}\,\ln t\,dt = -4(4-3\ln 2). \tag{3.5}$$

The further change of variables $t = s^b$ produces

$$\int_0^1 s^{b/a-1}\sqrt{1-s^b}\,\ln s\,ds = -\frac{a^2}{b^2}I(a). \tag{3.6}$$

For example, choosing $a=1$ and $b=2$ we obtain

$$\int_0^1 s\sqrt{1-s^2}\,\ln s\,ds = -\frac{1}{4}I(1) = \frac{1}{9}(3\ln 2 - 4). \tag{3.7}$$

• 4.241.10 This appears as entry 4.241.10 in [1].

4. A second family of examples

Several evaluations in [1] belong to the family

$$T_{i,j} = \int_0^\infty \frac{x^i e^{-jx}\,dx}{\sqrt{e^x-1}}. \tag{4.1}$$

The evaluation starts with the change of variables $t = e^{-x}$ that yields

$$T_{i,j} = (-1)^i \int_0^1 \frac{(\ln t)^i\, t^{j-1/2}}{\sqrt{1-t}}\,dt. \tag{4.2}$$

The power of the logarithm, for $i \in \mathbb{N}$, comes from differentiating

$$B(j+\tfrac{1}{2},\tfrac{1}{2}) = \int_0^1 t^{j-1/2}(1-t)^{-1/2}\,dt \tag{4.3}$$

with respect to j. We conclude that

$$T_{i,j} = (-1)^i\sqrt{\pi}\left(\frac{d}{dj}\right)^i \left[\frac{\Gamma(j+1/2)}{\Gamma(j+1)}\right]. \tag{4.4}$$

Using the duplication formula for the gamma function

$$\Gamma(x+1/2) = \frac{\sqrt{\pi}\Gamma(2x)}{2^{2x-1}\Gamma(x)} \tag{4.5}$$

we obtain

$$T_{i,j} = (-1)^i\pi\left(\frac{d}{dj}\right)^i \frac{\Gamma(2j)}{2^{2j-1}\,\Gamma(j+1)\Gamma(j)}. \tag{4.6}$$

In preparation for the case $j=0$ we write this as

$$T_{i,j} = (-1)^i\pi\left(\frac{d}{dj}\right)^i \frac{\Gamma(2j+1)}{2^{2j}\,\Gamma^2(j+1)}. \tag{4.7}$$

• Entry 3.452.1 states that

$$\int_0^\infty \frac{x\,dx}{\sqrt{e^x-1}} = 2\pi\ln 2. \tag{4.8}$$

In this case we have $i=1$ and $j=0$ so • 3.452.1

$$T_{1,0} = -\pi \left(\frac{d}{dj} \right) \frac{\Gamma(2j)}{2^{2j-1}\,\Gamma(j+1)\Gamma(j)} \tag{4.9}$$

evaluated at $j=0$. To avoid the singularity at $j=0$ we write

$$T_{1,0} = -\pi \left(\frac{d}{dj} \right) \frac{\Gamma(2j+1)}{2^{2j}\,\Gamma^2(j+1)}. \tag{4.10}$$

Logarithmic differentiation yields

$$T_{1,0} = -\pi \frac{\Gamma(2j+1)}{2^{2j}\Gamma^2(j+1)} \left(2\psi(2j+1) - 2\ln 2 - 2\psi(j+1) \right). \tag{4.11}$$

Replacing $j=0$ yields $T_{1,0} = 2\pi \ln 2$ as claimed.

• Entry 3.452.2 is the evaluation of $T_{2,0}$:

$$\int_0^\infty \frac{x^2\,dx}{\sqrt{e^x-1}} = \frac{\pi}{3} \left(12 \ln^2 + \pi^2 \right). \tag{4.12}$$

The identity (4.7) gives • 3.452.2

$$T_{2,0} = \pi \left(\frac{d}{dj} \right)^2 u(j) \tag{4.13}$$

where

$$u(j) = \frac{\Gamma(2j+1)}{2^{2j}\,\Gamma^2(j+1)}. \tag{4.14}$$

Observe that $u(0)=1$ and logarithmic differentiation yields

$$u'(j) = u(j) \left(2\psi(2j+1) - 2\ln 2 - 2\psi(j+1) \right). \tag{4.15}$$

Thus, $u'(0) = -2\ln 2$. One more differentiation produces

$$u''(j) = 2u'(j) \left(\psi(2j+1) - \psi(j+1) - \ln 2 \right) + 2u(j) \left(2\psi'(2j+1) - \psi'(j+1) \right).$$

Therefore $u''(0) = 4\ln^2 + 2\psi'(1)$. The result now follows from the special value

$$\psi'(1) = \frac{\pi^2}{6}. \tag{4.16}$$

• Entry 3.452.3 is the evaluation of $T_{1,1}$:

$$\int_0^\infty \frac{x\,e^{-x}\,dx}{\sqrt{e^x-1}} = \frac{\pi}{2} \left(2\ln^2 - 1 \right). \tag{4.17}$$

The identity (4.7) gives • 3.452.3

$$T_{1,1} = -\pi\, u'(1) \tag{4.18}$$

where $u(j)$ is as in (4.14). Observe that $u(1) = 1/2$ and logarithmic differentiation gives

$$u'(j) = 2u(j) \left(\psi(2j+1) - \ln 2 - \psi(j+1) \right). \tag{4.19}$$

We conclude that

$$T_{1,1} = -\pi(\psi(3) - \ln 2 - \psi(2)). \tag{4.20}$$

Using the special values

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \tag{4.21}$$

we obtain the result.

- Entry 3.452.4 is the evaluation of

$$\int_0^\infty \frac{x\, e^{-x}\, dx}{\sqrt{e^{2x}-1}} = 1 - \ln 2. \tag{4.22}$$

• 3.452.4 The change of variables $t = 2x$ shows that

$$\int_0^\infty \frac{x\, e^{-x}\, dx}{\sqrt{e^{2x}-1}} = \frac{1}{4} T_{1,\frac{1}{2}}. \tag{4.23}$$

The identity (4.7) gives

$$T_{1,\frac{1}{2}} = -\pi u'(1/2) \tag{4.24}$$

and from (4.19) we get

$$u'(\tfrac{1}{2}) = \frac{4}{\pi} \left(\psi(2) - \ln 2 - \psi(\tfrac{3}{2}) \right). \tag{4.25}$$

The special value

$$\psi(n + \tfrac{1}{2}) = -\gamma + 2 \left(\sum_{j=1}^{n} \frac{1}{2j-1} - \ln 2 \right) \tag{4.26}$$

gives

$$\psi(\tfrac{3}{2}) = -\gamma + 2 - 2\ln 2 \tag{4.27}$$

and the result follows from here.

- Entry 3.452.5 is the evaluation of

$$\int_0^\infty \frac{x\, e^{-2x}\, dx}{\sqrt{e^{x}-1}} = \frac{3\pi}{4} \left(\ln 2 - \frac{7}{12} \right). \tag{4.28}$$

• 3.452.5 This is $T_{1,2}$ and the evaluation follows from the values of $\psi(5)$ and $\psi(3)$ as before.

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