# A BINARY TREE REPRESENTATION FOR THE 2-ADIC VALUATION OF A SEQUENCE ARISING FROM A RATIONAL INTEGRAL

XINYU SUN AND VICTOR H. MOLL

Abstract. We analyze properties of the 2-adic valuation of an integer sequence that originates from an explicit evaluation of a quartic integral. We present a tree that encodes this valuation.

## 1. Introduction

The integral

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}, \tag{1.1}$$

with $a > -1$ is given by

$$N_{0,4}(a,m) = \frac{\pi}{2} \frac{P_m(a)}{[2(a+1)]^{m+1/2}} \tag{1.2}$$

where

$$P_m(a) = \sum_{l=0}^{m} d_{l,m} a^l \tag{1.3}$$

with

$$d_{l,m} = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}, \quad 0 \leq l \leq m. \tag{1.4}$$

The reader will find in [2] a survey of the different proofs of (1.2) and an introduction to the many issues involved in the evaluation of definite integrals in [6].

The study of combinatorial aspects of the sequence $d_l(m)$ was initiated in [3] where the authors show that $d_l(m)$ form a *unimodal* sequence, that is, there exists and index $l^*$ such that $d_{0,m} \leq \ldots \leq d_{l^*,m}$ and $d_{l^*,m} \geq \ldots \geq d_{m,m}$. The fact that $d_{l,m}$ satisfies the stronger condition of *logconcavity* $d_{l-1,m}d_{l+1,m} \leq d_{l,m}^2$ has been recently established in [5].

We consider here arithmetical properties of the sequence $d_{l,m}$. It is more convenient to analyze the auxiliary sequence

$$A_{l,m} = l!\, m!\, 2^{m+l} d_{l,m} = \frac{l!\, m!}{2^{m-l}} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l} \tag{1.5}$$

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for $m \in \mathbb{N}$ and $0 \leq l \leq m$. The integral (1.1) is then given explicitly as

$$N_{0,4}(a;m) = \frac{\pi}{\sqrt{2}\, m!\, (4(2a+1))^{m+1/2}} \sum_{l=0}^{m} A_{l,m} \frac{a^l}{l!}. \tag{1.6}$$

We present here a binary tree that encodes the 2-adic valuation of $A_{l,m}$. Recall that, for $x \in \mathbb{N}$, the 2-adic valuation $\nu_2(x)$ is the highest power of 2 that divides $x$. This is extended to $x = a/b \in \mathbb{Q}$ via $\nu_2(x) = \nu_2(a) - \nu_2(b)$.

Given $l \in \mathbb{N}$ we associate a tree $T(l)$, the *decision tree of* $l$, that provides a combinatorial interpretation of $\nu_2(A_{l,m})$. It has the following properties:

1) Aside from the labels on the vertices, $T(l)$ depends only on the odd part of $l$. Therefore it suffices to consider $l$ odd.

2) For $l$ odd, define $k^*(l) = \lfloor \log_2 l \rfloor$. The index $k^*$ is determined by $2^{k^*} < l < 2^{k^*+1}$.

3) The generations are labelled starting at 0; that is, the root is generation 0. For $0 \leq k \leq k^*$, the $k$-th generation consists of $2^k$ vertices. These form a complete binary tree.

4) A vertex with degree 1 is called *terminal.* The edge containing a terminal vertex is called a *terminal branch.* The $k^*$-th generation contains $2^{k^*+1} - l$ terminal vertices. The tree $T(l)$ has one more generation consisting of $2(l - 2^{k^*})$ terminal vertices.

5) Each terminal vertex of $T(l)$ has a *vertex constant* attached to it. These are given in Lemmas 2.8 and 2.10. Each non-terminal vertex has two *edge functions* attached to it.

The main results presented here is:

**Theorem 1.1.** *Let $l \in \mathbb{N}$. The data described above provides an explicit formula for the 2-adic valuation of the sequence $A_{l,m}$.*

The complete results are described in Section 2 and illustrated here for $l = 3$.

The sequence $\{\nu_2(A_{3,m}) : m \geq 1\}$ satisfies $\nu_2(A_{3,2m-1}) = \nu_2(A_{3,2m})$. Therefore the subsequence $A_{3,2m+1}$, denoted by $C_{3,m}$, contains all the 2-adic information of $A_{3,m}$.

For instance, at the first level in Figure 1 we have two edges with functions $2m$ and $2m-1$ and the first generation consists of two vertices, one of which is terminal with vertex constant 7. This vertex is adjacent to the terminal branch labeled $2m-1$.

This tree produces a formula for $\nu_2(C_{3,m})$ by the following mechanism: define

$$f_3(m) = \begin{cases} 7 + \nu_2\left(\frac{m+1}{2}\right) & \text{if } m \equiv 1 \bmod 2, \\ 9 + \nu_2\left(\frac{m}{4}\right) & \text{if } m \equiv 0 \bmod 4, \\ 9 + \nu_2\left(\frac{m+2}{4}\right) & \text{if } m \equiv 2 \bmod 4. \end{cases} \tag{1.7}$$

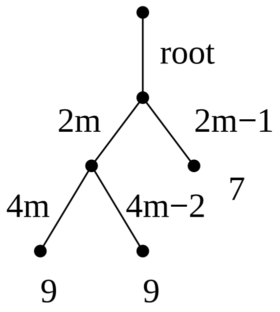

FIGURE 1. The decision tree for $l = 3$

There is one expression per terminal branch. The numbers 9, 9, 7 are the vertex constants of $T(3)$ and the arguments of $\nu_2$ in $f$ come from the branch labels. The tree now encodes the formula

$$\nu_2(C_{3,m}) = f_3(m), \text{ for } m \geq 1. \tag{1.8}$$

## 2. THE TREE

In this section we describe a binary tree that encodes the 2-adic valuation of the sequence $A_{l.m}$. This value is linked to that of the Pochhammer symbol

$$(a)_n := \begin{cases} a(a+1)(a+2)\cdots(a+n-1), & \text{for } n > 0 \\ 1, & \text{for } n = 0, \end{cases} \tag{2.1}$$

via the identity

$$\nu_2(A_{l,m}) = \nu_2((m+1-l)_{2l}) + l, \tag{2.2}$$

established in [1]. This is a generalization of the main result in [4], namely,

$$\nu_2(A_{1,m}) = \nu_2(m(m+1)) + 1. \tag{2.3}$$

The expression (2.2) can also be written as

$$\nu_2(A_{l,m}) = l + \sum_{j=-l+1}^{l} \nu_2(m+j). \tag{2.4}$$

To encode the information about $\nu_2(A_{l,m})$ we employ the notion of *simple sequences.*

**Definition 2.1.** A sequence $\{a_n : n \in \mathbb{N}\}$ is called $s$-simple if there exists a number $s$ such that, for each $t \in \{0, 1, 2, \cdots\}$, we have

$$a_{st+1} = a_{st+2} = \cdots = a_{s(t+1)}. \tag{2.5}$$

In pictorial terms, $s$-simple sequences are formed by blocks of length $s$ where they attain the same value. In [1] it is shown that, for fixed $l \in \mathbb{N}$, the sequence $\{\nu_2(A_{l,m}) : m \geq l\}$ is $2^{1+\nu_2(l)}-$simple. For instance,

$$\nu_2(A_{2,m}) = \{5, 5, 5, 5, 6, 6, 6, 6, 5, 5, 5, 5, 7, 7, 7, 7, 5, 5, 5, 5, \ldots\}. \tag{2.6}$$

is 4-simple.

**Definition 2.2.** Let $l \in \mathbb{N}$ be fixed. Define

$$C_{l,m} = A_{l,l+(m-1)\cdot 2^{1+\nu_2(l)}}, \tag{2.7}$$

so that the sequence $\{C_{l,m} : m \geq 1\}$ reduces each block of $A_{l,m}$ to a single point. In particular, for $l$ odd we have $C_{l,m} = A_{l,2(m-1)}$.

**The tree associated to** $l$. We associate to each index $l \in \mathbb{N}$ a tree by the following rule: start with a *root* vertex. This root is the 0-th generation of $T(l)$. To the root vertex we attach the sequence

$$\{\nu_2(C_{l,m}) : m \geq 1\} \tag{2.8}$$

and ask whether

$$\nu_2(C_{l,m}) - \nu_2(m) \tag{2.9}$$

is independent of $m$. If the answer is yes, we label the vertex $v_0$ with this constant value. This is the case for $l = 4$ as shown in Figure 2. If the answer is negative, we split the integers into classes modulo 2 and create a vertex for each class. These two classes are attached to two new vertices

$$v_1 \mapsto \{\nu_2(C_{l,2m-1}) : m \geq 1\}$$

and

$$v_2 \mapsto \{\nu_2(C_{l,2m}) : m \geq 1\}.$$

Each positive answer produces the end of the branch and each negative one yields two new branches that need to be tested. The process stops when there are no more vertices that need to be tested.

FIGURE 2. The decision tree for $l = 4$

**Note**. Assume the vertex $v$ corresponding to the sequence $\{2^k(m-1)+a : m \geq 1\}$ produces a negative answer. Then it splits in the next generation into two vertices corresponding to the sequences $\{2^{k+1}(m-1)+a : m \geq 1\}$ and $\{2^{k+1}(m-1)+2^k+a : m \geq 1\}$. For instance, in Figure 3, the vertex corresponding to $\{4m : m \geq 1\}$, that is not terminal, splits into $\{8m : m \geq 1\}$ and $\{8m+4 : m \geq 1\}$. These two edges lead to terminal vertices. Theorem 2.7 shows that this process ends in a finite number of steps.

Figure 2 shows the decision tree for $l = 4$ and Figure 1 the corresponding one for $l = 3$. In the latter figure the branches are labelled according to the arithmetic sequence they represent.

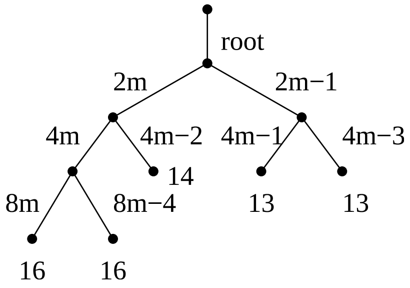

FIGURE 3. The decision tree for $l = 5$

At the second level we find the first appearance of a terminal vertex, namely the one corresponding the edge marked $2m-1$. Its vertex constant is 7, stating that

$$\nu_2(C_{3,2m-1}) = \nu_2(m) + 7. \tag{2.10}$$

The first step in the analysis of the tree $T(l)$ is to reduce it to the case where $l$ is odd.

**Theorem 2.3.** *The tree of an integer depends only upon its odd part, that is, for $b$ odd and $a \in \mathbb{N}$, we have*

$$T(b) = T(2^a\, b), \tag{2.11}$$

*up to relabel.*

The proof of this theorem is based on a relation of the 2-adic valuations of $C_{2l,m}$ and $C_{l,m}$. We establish first an auxiliary result for $A_{l,m}$.

**Lemma 2.4.** *Let $l$, $m \in \mathbb{N}$. Then*

$$\nu_2(A_{2l,2m}) = \nu_2(A_{l,m}) + 3l. \tag{2.12}$$

*Proof.* The result is equivalent to

$$\nu_2\left(a_l/a_{-l}\right) = 2l, \tag{2.13}$$

where

$$a_k = \frac{(2m+2k)!}{(m+k)!}. \tag{2.14}$$

This follows from

$$\nu_2\left(a_l/a_{-l}\right) = \sum_{k=1-l}^{l} \nu_2\left(a_k/a_{k-1}\right), \tag{2.15}$$

and $a_k/a_{k-1} = 2(2m+2k-1)$, so that each term in the sum (2.15) is equal to 1. □

**Corollary 2.5.** *Let $l$, $m \in \mathbb{N}$. Then*

$$\nu_2\left(C_{2l,m}\right) = \nu_2\left(C_{l,m}\right) + 3l. \tag{2.16}$$

*Proof.* The result follows from the identity

$$C_{2l,m} = A_{2l,2\left[l+(m-1)\cdot 2^{1+\nu_2(l)}\right]} \tag{2.17}$$

and (2.12). □

**Note**. Corollary 2.5 and the fact that the index $l$ is fixed yield the proof of Theorem 2.3.

From now on we assume $l \in \mathbb{N}$ is a fixed odd number. Consider now the sets

$$V_{k,a}^{l} := \left\{ \nu_2\left(C_{l,2^k(m-1)+a}\right) :\, m \in \mathbb{N} \right\} \tag{2.18}$$

for $k \in \mathbb{N}$ and $1 \leq a \leq 2^k$. Observe that, for fixed $k \in \mathbb{N}$ the $2^k$ sets $V_{k,a}^{l}$ contain all the information of the sequence $\{\nu_2\left(C_{l,m}\right) :\, m \geq 1\}$. For example, for $k=2$, we have

$$\begin{aligned} V_{2,1}^{l} &= \{\nu_2(C_{l,4m-3}) : m \in \mathbb{N}\}, \\ V_{2,2}^{l} &= \{\nu_2(C_{l,4m-2}) : m \in \mathbb{N}\}, \\ V_{2,3}^{l} &= \{\nu_2(C_{l,4m-1}) : m \in \mathbb{N}\}, \\ V_{2,4}^{l} &= \{\nu_2(C_{l,4m}) : m \in \mathbb{N}\}. \end{aligned}$$

These four sets correspond to the third generation in the tree shown in Figure 3. We also introduce the difference between $V_{k,a}^{l}$ and the basic sequence $\{\nu_2(m) : m \in \mathbb{N}\}$.

**Note**. The sets $V_{k,a}^{l}$ are attached to the vertices in the $k$-th generation of the decision tree $T(l)$. The terminal vertices of $T(l)$ are those corresponding to indices $a$ such that the set

$$S_{k,a}^{l} := \left\{ \nu_2\left(C_{l,2^k(m-1)+a}\right) - \nu_2(m) :\, m \in \mathbb{N} \right\} \tag{2.19}$$

reduces to a single value.

**Definition 2.6.** Let $k^*(l)$ be the first generation in the decision tree $T(l)$ that contains a terminal vertex. This is the minimal $k$ for which there exist an index $a$, in the range $1 \leq a \leq 2^k$, such that $V_{k,a}^{l}$ is a constant shift of the sequence $\{\nu_2(n) : n \in \mathbb{N}\}$.

**Theorem 2.7.** *Let $l \in \mathbb{N}$ be odd. Then $k^*(l) = \lfloor \log_2 l \rfloor$. The $k^*$−th generation contains $2^{k^*+1} - l$ terminal vertices. The tree $T(l)$ has one more generation consisting of $2(l-2^{k^*})$ terminal vertices. And these are the only terminal vertices.*

The proof is divided into a sequence of steps.

**Lemma 2.8.** *Let $l$ be an odd integer and define $k$ via $2^k < l < 2^{k+1}$. Then for $a$ in the range $1 \leq a \leq 2^{k+1} - l$ define $j_1(l,k,a) := -l + 2(1 + 2^k - a)$. Then*

$$\nu_2\left(C_{l,2^k(m-1)+a}\right) = \nu_2(m) + \gamma_1(l,k,a) \tag{2.20}$$

*with*

$$\gamma_1(l,k,a) = l + k + 1 + \nu_2\left((j_1 + l - 1)! \times (l - j_1)!\right). \tag{2.21}$$

*Therefore, the vertex corresponding to the index $a$ is a terminal vertex for the tree $T(l)$ with vertex constant $\gamma_1(l,k,a)$.*

*Proof.* We have

$$\begin{aligned} (2.22)\quad \nu_2\left(C_{l,2^k(m-1)+a}\right) &= \nu_2\left(A_{l,l+2(2^k(m-1)+a-1)}\right) \\ &= l + \sum_{j=-l+1}^{l} \nu_2\left(l+2(2^k(m-1)+a-1)+j\right). \end{aligned}$$

The bounds on $a$ imply that $2-l \leq j_1 \leq 2^{k+1}-l$ showing that $j_1$ is in the range of summation. Morever it isolates the term $2^{k+1}m$; that is, (2.22) can be computed as

$$\nu_2\left(C_{l,2^k(m-1)+a}\right) = l + \sum_{b=1}^{j_1+l-1} \nu_2(2^{k+1}m-b) + k + 1 + \nu_2(m) + \sum_{b=1}^{l-j_1} \nu_2(2^{k+1}m+b).$$

In the first sum we have $b \leq j_1+l-1 = 1-2a+2^{k+1} < 2^{k+1}$, and in the second one $b \leq l-j_1 = 2\left(l-1+a-2^k)\right) < 2^{k+1}$, by the choice of the upper bound on $a$. We conclude that

$$\nu_2\left(C_{l,2^k(m-1)+a}\right) = \nu_2(m) + l + k + 1 + \sum_{b=1}^{j_1+l-1} \nu_2(b) + \sum_{b=1}^{l-j_1} \nu_2(b).$$

This is the stated result. □

**Lemma 2.9.** *Let $k$ and $l$ be defined as above, then*

$$(2.23)\qquad \nu_2\left(A_{l,2^{k+1}m+a}\right) = \nu_2\left(A_{l,2^{k+1}m-a-1}\right),$$

*for any $m \geq 1$ and $0 \leq a < 2^{k+1}-l$.*

*Proof.* Since $2^k < a+l < 2^{k+1}$, $0 \leq a \leq 2^k$, and $-2^{k+1} < a-l < 0$. Therefore,

$$\begin{aligned} \nu_2\left(A_{l,2^{k+1}m+a}\right) &= l + \sum_{j=-l+1}^{l} \nu_2\left(2^{k+1}m+a+j\right) \\ &= l + \sum_{j=-l}^{l-1} \nu_2\left(2^{k+1}m-a-j\right) \\ &= \nu_2\left(A_{l,2^{k+1}m-a-1}\right). \end{aligned}$$

□

Now since the sets $\{2^{k+1}m \pm a \,|\, 0 \leq a < 2^{k+1}+2^k-l, m \geq 1\}$ and $\{2^{k+2}m \pm b \,|\, 2^{k+1}-l < b < l, m \geq 1\}$ partition the set $\{a \,|\, a \geq l\}$, to prove the second half of Theorem 2.7, we only need to show the following.

**Lemma 2.10.** *Let $k$ and $l$ be defined as above, then for $a$ in the range $2^{k+1}-l < a \leq 2^{k+1}$ define $j_2(l,k,a) := -l+2(1+2^{k+1}-a)$. Then*

$$(2.24)\qquad \nu_2\left(C_{l,2^k(m-1)+a}\right) = \nu_2(m) + \gamma_2(l,k,a)$$

*with*

$$(2.25)\qquad \gamma_2(l,k,a) = l+k+2+\nu_2\left((j_2+l-1)! \times (l-j_2)!\right).$$

*Therefore, the vertex corresponding to the index $a$ is a terminal vertex for the tree $T(l)$ with vertex constant $\gamma_2(l,k,a)$.*

*Proof.* The proof is the same as that of Lemma 2.8, and thus omitted. □

**Example 2.11.** In the case $l = 3$ we can take $k = 1$. On the higher level, the restrictions on the parameter $a$ imply that must have $a = 1$. A direct calculation shows that $j_1(3,1,1) = 1$ and $\gamma_1(3,1,1) = 7$. For the bottom two vertices, $a = 2, 4$; and we have $j_2(3,1,2) = 3, \gamma_2(3,1,2) = 9$; while $j_2(3,1,4) = -1, \gamma_2(3,1,4) = 9$. This confirms the data on Figure 1.

**Example 2.12.** For $l = 5$, the theorem predicts three terminal vertices at the level $k = 2$, corresponding to the values $a = 1, 2, 3$. This confirms Figure 3 with terminal values given by $\gamma_1(5,2,1) = \gamma_1(5,2,3) = 13$ and $\gamma_1(5,2,2) = 14$. Similar results can be drawn for the level $k = 3$. As before, the tree produces an explicit formula for the 2-adic valuation of $C_{5,m}$. Indeed, define

$$f_5(m) = \begin{cases} 14 + \nu_2\left(\frac{m+2}{4}\right) & \text{if } m \equiv 2 \bmod 4, \\ 13 + \nu_2\left(\frac{m+1}{4}\right) & \text{if } m \equiv 3 \bmod 4, \\ 13 + \nu_2\left(\frac{m+3}{4}\right) & \text{if } m \equiv 1 \bmod 4, \\ 16 + \nu_2\left(\frac{m}{8}\right) & \text{if } m \equiv 0 \bmod 8, \\ 16 + \nu_2\left(\frac{m+4}{8}\right) & \text{if } m \equiv 4 \bmod 8. \end{cases} \tag{2.26}$$

then,

$$\nu_2(C_{5,m}) = f_5(m). \tag{2.27}$$

To finish the proof of Theorem 2.7, we need to establish

**Lemma 2.13.** *There are no terminal vertices of level less than $k$.*

*Proof.* The value of a vertex on the level $u < k$ is obtained from $\nu_2\left(C_{l,2^u(m-1)+a}\right)$. The proof of Lemma 2.8, shows that

$$\nu_2\left(C_{l,2^u(m-1)+a}\right) = \sum_{i=0}^{v} \nu_2(m+i) + c,$$

for some constants $v > 0$ and $c$. The next lemma proves that this cannot happen. □

**Lemma 2.14.** *If*

$$\sum_{i=0}^{a} \nu_2(m+i) = \nu_2(m+b) + c \tag{2.28}$$

*for all $m \geq 1$, and some constants $a, b, c$, then $a = b = c = 0$.*

*Proof.* Suppose the lemma is not true and $b > a$. Choose $m$ such that $m+b = 2^u$ for some $u$, then $\sum_{i=0}^{a} \nu_2(m+i) = \nu_2((b-a)\cdots b)$. Therefore $c = \nu_2((b-a)\cdots b) - u$. Similarly choose $m$ such that $m+b = 2^{u+1}$, and conclude that $c = \nu_2((b-a)\cdots b) - u - 1$. This is a contradiction. The proof to the other two cases where $0 \leq b \leq a$ and $b < 0$ are similar, and thus omitted. □

The final figure shows how to produce the trees corresponding to $l$ odd. First determine $n$ by $2^n < l < 2^{n+1}$ and form a complete binary tree $T$ where the last level has $2^n$ vertices. Now from $T$ branch an odd number of vertices that yields the decision trees $T(l)$. Figure 4 shows the four trees corresponding to the odd indices $l$ in the range $8 < l < 16$.

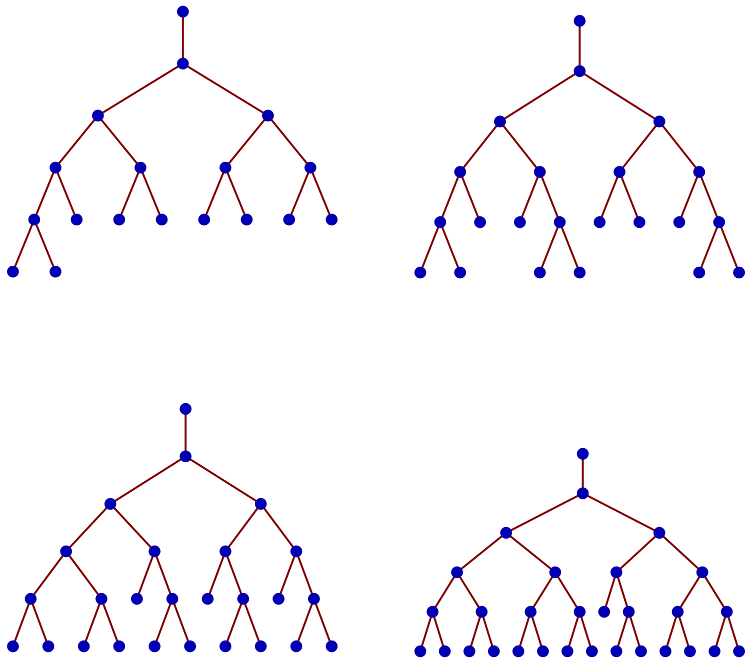

FIGURE 4. The trees for $l$ odd between 8 and 16

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DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
*E-mail address*: `xsun1@tulane.edu`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
*E-mail address*: `vhm@math.tulane.edu`