THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 21: TRIGONOMETRIC FUNCTIONS. PRELIMINARY VERSION: LAST UPDATE SEPTEMBER 19, 2006.

VICTOR H. MOLL

Abstract. We present a systematic derivation of some of the definite integrals in the classical table of Gradshteyn and Rhyzik that involve trigonometric functions.

1. Introduction

The table of integrals [1] contains many evaluations that involve the standard trigonometric functions. Our goal is to present them in a systematic manner.

2. Some elementary observations about symmetry

Consider the integral

$$J_{i,j} := \int_0^{2\pi} \sin^i t \, \cos^j t \, dt. \tag{2.1}$$

The change of variables $v = 2\pi - t$ produces

$$J_{i,j} = (-1)^i J_{i,j}. \tag{2.2}$$

Therefore,

$$J_{i,j} = 0, \text{ if } i \text{ is odd.} \tag{2.3}$$

We now consider the value of

$$S_m(a,b) = \int_0^{2\pi} (a \, \sin x + b \, \cos x)^m \, dx. \tag{2.4}$$

The change of variables $u = 2\pi - x$ yields

$$S_m(a,b) = \int_0^{2\pi} (-a \, \sin x + b \, \cos x)^m \, dx. \tag{2.5}$$

Adding these two equations produces

$$2S_m(a,b) = \sum_{k=0}^{m} \binom{m}{k} (1+(-1)^k) a^k b^{m-k} \int_0^{2\pi} \sin^k x \, \cos^{m-k} x \, dx.$$

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Assume now that m is odd, say $m = 2n+1$. Then k must be even, say $k = 2j$, with $0 \leq j \leq n$. We obtain

$$\begin{aligned} (2.6) \qquad 2S_m(a,b) &= \sum_{j=0}^{n} \binom{2n+1}{2j} a^{2j} b^{2n+1-2j} \int_0^{2\pi} \sin^{2j} \cos^{2n+1-2j} x \, dx \\ &= \sum_{j=0}^{n} \binom{2n+1}{2j} a^{2j} b^{2n+1-2j} J_{2j,2n+1-2j}. \end{aligned}$$

Now use periodicity and symmetry to write

$$\begin{aligned} \int_0^{2\pi} \sin^{2j} x \, \cos^{2n+1-2j} x \, dx &= \int_{-\pi}^{\pi} \sin^{2j} x \, \cos^{2n+1-2j} x \, dx \\ &= 2 \int_0^{\pi} \sin^{2j} x \, \cos^{2n+1-2j} x \, dx. \end{aligned}$$

Now observe that the change of variables $u = \pi - x$ yields

$$\begin{aligned} \int_{\pi/2}^{\pi} \sin^{2j} x \, \cos^{2n+1-2j} x \, dx &= (-1)^{2n+1-2j} \int_0^{\pi/2} \sin^{2j} u \, \cos^{2n+1-2j} u \, du \\ &= -\int_0^{\pi/2} \sin^{2j} x \, \cos^{2n+1-2j} x \, dx. \end{aligned}$$

Therefore

$$(2.7) \qquad J_{2j,2n+1-2j} = \int_0^{2\pi} \sin^{2j} x \, \cos^{2n+1-2j} x \, dx = 0.$$

From (2.6) we conclude that

$$(2.8) \qquad S_m(a,b) = 0, \text{ if } m \text{ is odd.}$$

• 3.661.1

This appears as 3.661.1 in [1].

The case m even, say $m = 2n$ is treated in a similar way. Symmetry produces

$$(2.9) \qquad S_m(a,b) = 4 \sum_{j=0}^{n} \binom{2n}{2j} a^{2j} b^{2n-2j} \int_0^{\pi/2} \sin^{2j} x \, \cos^{2n-2j} x \, dx.$$

The integral above is expressed in terms of the beta function

$$(2.10) \qquad B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} \, dt$$

as

$$(2.11) \qquad \int_0^{\pi/2} \sin^{2j} x \, \cos^{2n-2j} x \, dx = \frac{1}{2} B\left(j + \tfrac{1}{2}, n - j + \tfrac{1}{2}\right).$$

Using the relation $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, and the special value

$$(2.12) \qquad \Gamma\left(k + \tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2k}} \frac{(2k)!}{k!},$$

we obtain

$$(2.13) \qquad S_{2n}(a,b) = \frac{(2n)! \, \pi}{2^{2n-1} \, n!} \sum_{k=0}^{n} \frac{a^{2k} \, b^{2n-2k}}{k! \, (n-k)!}.$$

This reduces to

$$S_{2n}(a,b) = \frac{2\pi}{2^{2n}} \binom{2n}{n} (a^2+b^2)^n. \tag{2.14}$$

• 3.661.2

This appears as 3.661.2.

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References

[1] I.S. Gradshteyn and I.M. Rhyzik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@math.tulane.edu