THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 16: WHITTAKER FUNCTIONS. PRELIMINARY VERSION: LAST UPDATE SEPTEMBER 11, 2006.

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Abstract. We present a systematic derivation of some of the definite integrals in the classical table of Gradshteyn and Rhyzik that can be reduced to the Whittaker function.

1. Introduction

The table of integrals [1] contains some evaluations that can be derived from the *Whittaker functions*, defined by

$$M_{a,b}(z) = X_{a,b}(z) \int_{-1}^{1} (1+t)^{b-a-1/2}(1-t)^{b+a-1/2} e^{zt/2}\, dt, \tag{1.1}$$

where

$$X_{a,b}(z) = \frac{z^{b+1/2}}{2^{2b}} B(a+b+\tfrac{1}{2}, b-a+\tfrac{1}{2}), \tag{1.2}$$

and also

$$W_{a,b}(z) = \frac{z^{b+1/2}e^{-z/2}}{\Gamma(b-a+1/2)} \int_0^\infty e^{-zt} t^{b-a-1/2}(1+t)^{b+a-1/2}\, dt. \tag{1.3}$$

Our goal is to present in a systematic manner, the evaluations appearing in the classical table of Gradshteyn and Rhyzik [1], that involve this function.

2. Some elementary changes of variables

The change of variables $t = 2x-1$ transforms the interval $[-1,1]$ onto $[0,1]$. We obtain

$$M_{a,b}(z) = X_{a,b}(z)e^{-z/2} \int_0^1 x^{b-a-1/2}(1-x)^{b+a-1/2} e^{xz}\, dx.$$

The change of variables $x = e^{-t}$ now produces

$$M_{a,b}(z) = X_{a,b}(z)e^{-z/2} \int_0^\infty (1-e^{-t})^{b+a-1/2} \text{exp}(ze^{-t} - t(b+a-1/2))\, dt.$$

The choice $\nu = b+a+1/2$ and $\mu = b-a+1/2$ produces

$$\int_0^\infty (1-e^{-t})^{\nu-1} \text{exp}(ze^{-t} - \mu t)\, dt = z^{-(\mu+\nu)/2} B(\mu,\nu) e^{z/2} M_{(\nu-\mu)/2,(\nu+\mu-1)/2}(z).$$

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This appears as 3.331.3 in [1]. • 3.331.3

The change of variables $t = e^x - 1$ converts (1.3) into

$$W_{a,b}(z) = \frac{z^{b+1/2}e^{z/2}}{\Gamma(b-a+1/2)} \int_0^\infty \exp(-ze^x + 2bx)(1-e^{-x})^{b-a-1/2}\, dx.$$

Using the notation $\mu = -2b$ and $\nu = b - a + 1/2$, so that $a = (1-\mu-2\nu)/2$ and $b = -\mu/2$, we obtain 3.331.4: • 3.331.4

$$\int_0^\infty \exp(-ze^x - \mu x)(1-e^{-x})^{\nu-1}\, dx = \Gamma(\nu)e^{-z/2}z^{(\mu-1)/2}W_{(1-\mu-2\nu)/2,-\mu/2}(z).$$

Writing $p = b - a - 1/2$ and $q = b + a - 1/2$, so that $a = (q-p)/2$ and $b = (p+q+1)/2$ converts (1.3) into

$$\int_0^\infty e^{-zt}t^p(1+t)^q\, dt = \frac{\Gamma(p+1)e^{z/2}}{z^{(p+q+2)/2}}\, W_{(q-p)/2,(p+q+1)/2}. \tag{2.1}$$

The change of variables $s = 1/t$ yields

$$\int_0^\infty e^{-z/s}s^{-p-q-2}(1+s)^q ds = \frac{\Gamma(p+1)e^{z/2}}{z^{(p+q+2)/2}}\, W_{(q-p)/2,(p+q+1)/2}(z).$$

The further change of variables $s = e^w - 1$ now yields

$$\int_0^\infty \exp\left[-\frac{z}{e^w-1} + (q+1)w\right](e^w-1)^{-p-q-2}\, dw = \frac{\Gamma(p+1)e^{z/2}}{z^{(p+q+2)/2}}\, W_{(q-p)/2,(p+q+1)/2}(z).$$

Now relabel the parameters by letting $\mu = -q - 1$ and $\nu - -p - q - 1$. We obtain

$$\int_0^\infty \exp\left[-\frac{z}{e^w-1} - \mu w\right](e^w-1)^{\nu-1}\, dw = \frac{\Gamma(\mu-\nu+1)e^{z/2}}{z^{(1-\nu)/2}}\, W_{\frac{\nu-2\mu-1}{2},-\frac{\nu}{2}}(z).$$

This appears as 3.334 in [1]. • 3.334

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References

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