THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 11: EVALUATION USING THE RIEMANN ZETA FUNCTION. PRELIMINARY VERSION: LAST UPDATE SEPTEMBER 6, 2006.

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Abstract. The table of Gradshteyn and Rhyzik contains many integrals that can be evaluated using the Riemann zeta function. Some examples are discussed.

1. Introduction

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.1}$$

The series converges for $\text{Re}\, s > 1$. The identity

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = (2^{1-s} - 1)\zeta(s) \tag{1.2}$$

is easy to establish. Start with

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} &= \sum_{m=1}^{\infty} \frac{1}{(2m)^s} - \sum_{m=1}^{\infty} \frac{1}{(2m-1)^s} \\
&= 2^{-s} \sum_{m=1}^{\infty} \frac{1}{m^s} - \left(\sum_{m=1}^{\infty} \frac{1}{m^s} - \sum_{m=1}^{\infty} \frac{1}{(2m)^s} \right).
\end{aligned}$$

The identity (1.2) follows from here.

Special values of this function are difficult. The simplest ones are

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \tag{1.3}$$

where B_{2n} is the *Bernoulli* number given by the generating function

$$\frac{u}{e^u - 1} = \sum_{k=0}^{\infty} B_k \frac{u^k}{k!}. \tag{1.4}$$

2. A first integral representation

• 3.411.1 The table [1] contains as 3.411.1 the representation

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$$\int_0^\infty \frac{x^{s-1}\,dx}{e^{px}-1} = \frac{\Gamma(s)}{p^s}\zeta(s). \tag{2.1}$$

The gamma function is defined by

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt. \tag{2.2}$$

To prove (2.1), let $t = sx$ to obtain

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^{px}-1} = p^{-s}\int_0^\infty \frac{t^{s-1}\,dt}{e^t-1}. \tag{2.3}$$

Now expand the integrand as

$$\frac{1}{e^t-1} = \frac{e^{-t}}{1-e^{-t}} = \sum_{k=0}^\infty e^{-(k+1)t}. \tag{2.4}$$

Therefore

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^{px}-1} = p^{-s}\sum_{k=0}^\infty \int_0^\infty t^{s-1}e^{-(1+k)t}\,dt. \tag{2.5}$$

The change of variables $y = (1+k)$ yields the result.

The special case (1.3) produces the evaluation • 3.411.2

$$\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}-1} = (-1)^{n-1}\left(\frac{2\pi}{p}\right)^{2n}\frac{B_{2n}}{4n}, \tag{2.6}$$

given as 3.411.2 in [1].

The formula 3.411.3 is similar to (2.1), only uses the alternating zeta function. It states that • 3.411.3

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^{px}+1} = \frac{(1-2^{1-s})\Gamma(s)}{p^s}\zeta(s). \tag{2.7}$$

To prove this, expand the integrand to get

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^{px}+1} = \frac{p^{-s}}{\Gamma(s)}\sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^s}, \tag{2.8}$$

and now use (1.2). The special case $s = 2n$, using (1.3), produces 3.411.4 in [1]: • 3.411.4

$$\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}+1} = (-1)^{n-1}(1-2^{1-2n})\left(\frac{2\pi}{p}\right)^{2n}\frac{B_{2n}}{4n}. \tag{2.9}$$

The case $s = 1$ has to be treated differently because of the pole of $\zeta(s)$. In this case we obtain • 3.311.1

$$\int_0^\infty \frac{dx}{1+e^x} = \frac{\ln 2}{p}. \tag{2.10}$$

This is an elementary integral that can be evaluated via $t = e^x$. Details appear in [2].

3. An exponential scale

The change of variables $x = e^{-t}/p$ converts (2.1) into 3.333.1:

$$\int_{-\infty}^{\infty} \frac{e^{-st}\,dt}{\exp(e^{-t})-1} = \Gamma(s)\zeta(s). \tag{3.1}$$

• 3.333.1 Similarly, $x = e^{-t}/p$ yields from (2.7) the evaluation of 3.333.2:

• 3.333.2

$$\int_{-\infty}^{\infty} \frac{e^{-st}\,dt}{\exp(e^{-t})+1} = (1-2^{1-s})\Gamma(s)\zeta(s). \tag{3.2}$$

The case $s = 1$ gives

$$\int_{-\infty}^{\infty} \frac{e^{-t}\,dt}{\exp(e^{-t})+1} = \log 2, \tag{3.3}$$

by using the change of variables $y = e^{-t}$.

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