MATH 115, FALL SEMESTER 2008

Exam 2

Name (print): ______________________________

INSTRUCTIONS

(1) Calculators are allowed.
(2) Present your solutions in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

QN	PTS
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
TOTAL	

(1) Let $f(x) = e^x$ and $g(x) = x + 2$.

(a) (5 points) Evaluate $f(g(0))$ and $g(f(1))$

Solution.

$$f(g(0)) = f(2) = e^2$$
$$g(f(1)) = g(e) = e + 2$$

□

(b) (5 points) How do you obtain the graph of $g(f(x-3))$ from that of $f(x)$?

Solution. By moving the graph of $f(x)$ to the right by 3 units, and up by 2 units. □

(2) (8 points) The position of an object moving along the x–axis at time t is $x(t) = t^2 - 3t + 10$. An external force is applied to the object. The magnitude of the force F is inversely proportional to the position of the object. When $t = 1$, $F = 4$. Write the magnitude of the force as a function of time.

Solution. Since the magnitude of the force F is inversely proportional to the position of the object, $F = \frac{k}{x(t)}$.

When $t = 1$, $x(1) = 8$, and $F = \frac{k}{x(1)} = 4$. So the constant of proportionality is 32. Therefore

$$F = \frac{32}{x(t)} = \frac{32}{(t^2 - 3t + 10)}$$

□

(3) Given the function below

x	1	2	3	4	5
$f(x)$	10	7	5	4	3.5

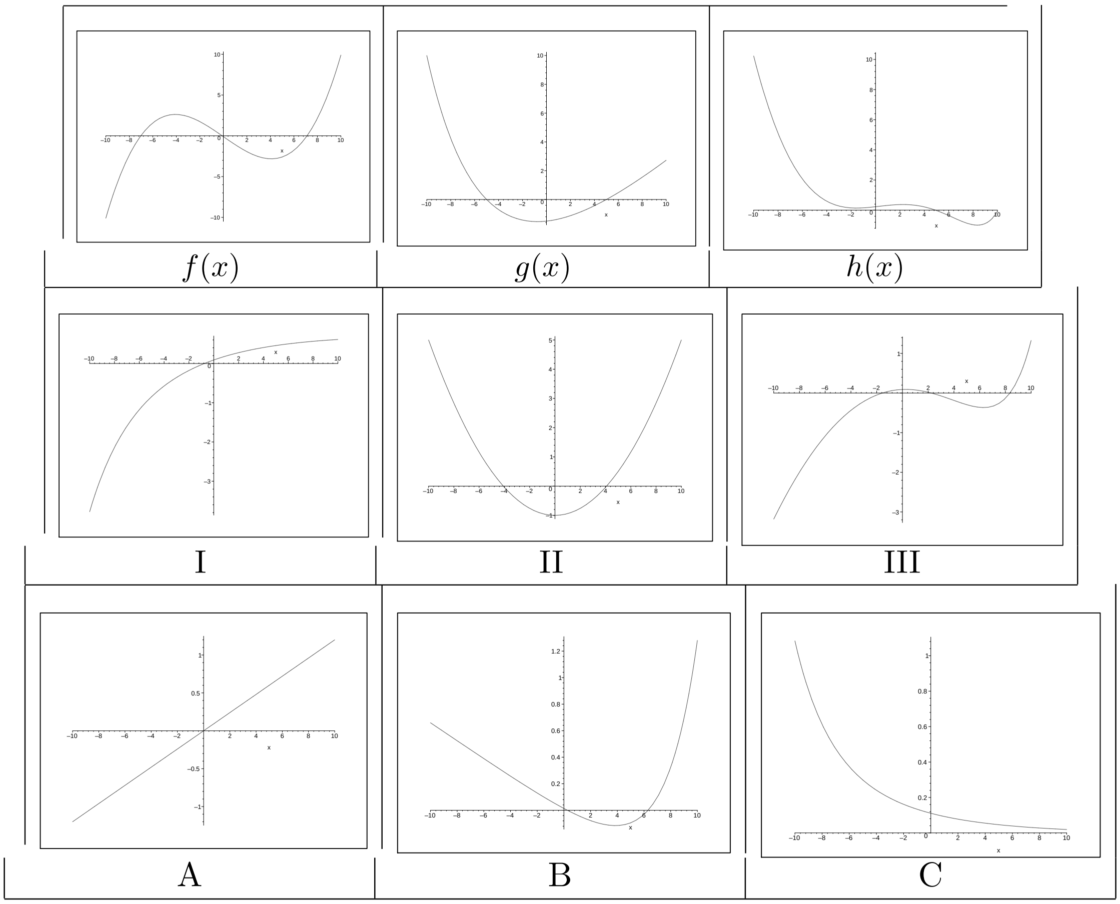
(a) (5 points) Is the derivative of the function positive or negative? Explain clearly.

Solution. Negative, because the function is decreasing. □

(b) (5 points) Is the second derivative of the function positive or negative? Explain clearly.

Solution. Positive, because the derivative is increasing. You can also say it is because the graph is concave up. □

(4) (9 points) Match the functions $f(x), g(x), h(x)$ with their derivatives (I, II, III), and second derivatives (A, B, C).



Solution.

$$f(x) \longrightarrow II \longrightarrow A$$
$$g(x) \longrightarrow I \longrightarrow C$$
$$h(x) \longrightarrow III \longrightarrow B$$

□

(5) (8 points) Find all the intervals on which the function is continuous. Explain clearly.

$$\frac{(x-2)(x-5)}{(x-1)(x+3)}$$

Solution. Since the denominator is 0 at $x = 1, -3$, the function is discontinuous at those points. So the function is continuous on $(-\infty, -3)$, $(-3, 1)$, and $(1, \infty)$. □

(6) A company makes machines. The cost function is $C(q) = q^2 - q + 1000$, and the revenue function is $R(q) = 100q - \sqrt{q}$. (Hint: $\sqrt{25} = 5, \sqrt{64} = 8$)

(a) (8 points) What is the profit if the company makes 25 machines? How about 64 machines?

Solution. The profit for making 25 machines is $R(25) - C(25) = 100 \cdot 25 - \sqrt{25} - (25^2 - 25 + 1000) = 895$.
The profit for making 64 machines is $R(25) - C(25) = 100 \cdot 64 - \sqrt{64} - (64^2 - 64 + 1000) = 1360$. □

(b) (8 points) Should the company make the 25^{th} machine? How about the 64^{th}? Explain clearly.

Solution. The marginal cost is $MC(q) = C'(q) = 2q - 1$.
The marginal revenue is $MR(q) = R'(q) = 100 - \frac{1}{2\sqrt{q}}$;
The marginal profit when the production level is 25 is

$$MR(64) - MC(64) = 100 - \frac{1}{10} - (2 \cdot 25 - 1) = \frac{509}{10}.$$

Therefore the company makes money when making the 25^{th} machine. It should do it.
The marginal profit when the production level is 64 is

$$MR(64) - MC(64) = 100 - \frac{1}{16} - (2 \cdot 64 - 1) = -\frac{431}{16}.$$

Therefore the company loses money when making the 64^{th} machine. It should not do it. □

(7) Given that $f(20) = 15$, $f'(20) = 2$, and $f''(x) < 0$ for all x, answer the following questions.

(a) (8 points) Estimate the value of $f(23)$.

Solution. $f(32) \approx f(20) + f'(20)(23 - 20) = 15 + 2 \cdot 3 = 21$. □

(b) (8 points) Is it possible that $f(30) = 35$?

Solution. Since the linear approximation of $f(20) + f'(20)(30 - 20) = 15 + 2 \cdot 10 = 35$. But since $f''(x) < 0$, the function is concave down, so the curve is below the tangent line, which means $f(30) < 35$. □

(8) (8 points) Use the definition of derivative to find $f'(3)$, where $f(x) = \frac{1}{x-1}$. No credit for any other method.

Solution.

$$f'(x) = \lim_{h\longrightarrow 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\longrightarrow 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} = \lim_{h\longrightarrow 0} \frac{\frac{(x-1)-(x+h-1)}{(x+h-1)(x-1)}}{h} = \lim_{h\longrightarrow 0} \frac{\frac{-h}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h\longrightarrow 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}$$

So

$$f'(3) = \frac{-1}{(3-1)^2} = -\frac{1}{4}$$

□

(9) Evaluate

(a) (5 points) $\frac{d}{dx}\left(\frac{x^2-5+7x^{1.3}}{\sqrt{x^3}}\right)$

Solution.

$$\frac{d}{dx}\left(\frac{x^2-5+7x^{1.3}}{\sqrt{x^3}}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}} - 5x^{-\frac{3}{2}} + 7x^{-0.2}\right)$$

$$= \frac{1}{2}x^{\frac{1}{2}-1} - 5\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1} + 7(-0.2)x^{-0.2-1} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{15}{2}x^{-\frac{5}{2}} - 1.4x^{-1.2}$$

□

(b) (5 points) $\lim_{t\to 2} t^2e^t - 3t5^t$

Solution.

$$\lim_{t\to 2} t^2e^t - 3t5^t = 2^2e^2 - 3\cdot 2\cdot 5^2 = 4e^2 - 150$$

□

(10) Given that $f(x) = \pi x^e + \pi^e$, answer the following questions.

(a) (5 points) Find $f'(x)$.

Solution.

$$f'(x) = \pi e x^{e-1}$$

□

(b) (5 points) Find the equation of the tangent line to $f(x)$ at the point where $x = 2$.

Solution.

$$f(2) = \pi 2^e + \pi^e$$
$$f'(2) = \pi e 2^{e-1}$$

So the equation is

$$y - (\pi 2^e + \pi^e) = \pi e 2^{e-1}(x - 2)$$

□