MATH 122, FALL SEMESTER 2008

Exam 1

Name (print): ____________________________

INSTRUCTIONS

(1) Calculators are NOT allowed.
(2) Present your solutions in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

QN	PTS
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
TOTAL	

(1) (10 points) Find the area between the curves $y = x$ and $y = x^2$ for $x \in [0, 2]$.

Solution.

$$A = \int_0^2 |f(x) - g(x)| dx = \int_0^1 (x - x^2) dx + \int_1^2 (x^2 - x) dx = \left(\frac{x}{2} - \frac{x^3}{3}\right)\Big|_0^1 + \left(\frac{x^3}{3} - \frac{x}{2}\right)\Big|_1^2 = \frac{1}{2} - \frac{1}{3} + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2} = 1$$

□

(2) (10 points) Find the volume of the solid obtained by rotating the region bounded by $y = x$, $y = x^2$, and $x = 2$ about $y = 5$.

Solution. The solid has the shape of a washer, whose center is $y = 5$, inner radius is $5 - x^2$ and outer radius is $5 - x$, therefore the area of the cross-section is $\pi(5 - x)^2 - \pi(5 - x^2)^2$.

$$V = \int_1^2 \pi \left((5 - x)^2 - (5 - x^2)^2\right) dx = \pi \int_1^2 \left(-10x + 11x^2 - x^4\right) dx = \frac{67}{15}\pi$$

□

(3) (10 points) A particle is moving along the x–axis starting from the origin. Its *velocity* is $t - 2$ ft/sec at time t. A force of $3t$ pounds is applied to the particle. Find the work done when $t = 2$.

Solution. Let the position of the particle ibe $x(t)$, then $dx = (t - 2)dt$.

Assume the particle travelling from point a to b. The work done is

$$\int_a^b F(x(t))dx = \int_0^2 3t(t - 2)dt = \int_0^2 (3t^2 - 6)dt = t^3 - 3t\big|_0^2 = 8 - 6 = 2(ft \cdot lb)$$

□

(4) (9 points) $\int x \tan^2 x dx$

Solution.

$$\begin{aligned}
\int x \tan^2 x dx &= \int x(\sec^2 x - 1)dx \\
&= \int x \sec^2 x dx - \int x dx \\
&= \int x d \tan x - \frac{x^2}{2} \\
&= x \tan x - \int \tan x dx - \frac{x^2}{2} \\
&= x \tan x + \ln|\cos x| - \frac{x^2}{2} + C
\end{aligned}$$

□

(5) (9 points) $\int \cos^3 x \sin^2 x dx$

Solution.

$$\begin{aligned}
\int \cos^3 x \sin^2 x dx &= \int \cos^2 x \sin^2 x d \sin x \\
&= \int (1 - \sin^2 x) \sin^2 x d \sin x \\
&= \int \sin^2 x - \sin^4 x d \sin x \\
&= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C
\end{aligned}$$

□

(6) (9 points) $\int \frac{\ln\sqrt{x}}{\sqrt[3]{x}}dx$

Solution.

$$\begin{aligned}\int \frac{\ln\sqrt{x}}{\sqrt[3]{x}}dx &= \int \ln\sqrt{x}\frac{3}{2}dx^{\frac{2}{3}} \\ &= \ln\sqrt{x}\frac{3}{2}x^{\frac{2}{3}} - \frac{3}{2}\int x^{\frac{2}{3}}d\ln\sqrt{x} \\ &= \ln\sqrt{x}\frac{3}{2}x^{\frac{2}{3}} - \frac{3}{2}\int x^{\frac{2}{3}}\frac{1}{\sqrt{x}}\frac{1}{2\sqrt{x}}dx \\ &= \ln\sqrt{x}\frac{3}{2}x^{\frac{2}{3}} - \frac{3}{4}\int x^{-\frac{1}{3}}dx \\ &= \ln\sqrt{x}\frac{3}{2}x^{\frac{2}{3}} - \frac{9}{8}x^{\frac{2}{3}} + C\end{aligned}$$

□

(7) (9 points) $\int_0^{2\pi} x^2 \sin x dx$

Solution.

$$\begin{aligned}\int_0^{2\pi} x^2 \sin x dx &= \int_0^{2\pi} -x^2 d\cos x \\ &= -x^2\cos x\Big|_0^{2\pi} + \int_0^{2\pi}\cos x dx^2 \\ &= -x^2\cos x\Big|_0^{2\pi} + \int_0^{2\pi} 2x\cos x dx \\ &= -x^2\cos x\Big|_0^{2\pi} + \int_0^{2\pi} 2x d\sin x \\ &= -x^2\cos x\Big|_0^{2\pi} + 2x\sin x\Big|_0^{2\pi} - \int_0^{2\pi} 2\sin x dx \\ &= -x^2\cos x\Big|_0^{2\pi} + 2x\sin x\Big|_0^{2\pi} + 2\cos x\Big|_0^{2\pi} \\ &= -4\pi^2\end{aligned}$$

□

(8) (9 points) $\int e^{2x}\cos x dx$

Solution.

$$\begin{aligned}\int e^{2x}\cos x dx &= \int e^{2x} d\sin x \\ &= e^{2x}\sin x - \int \sin x de^{2x} \\ &= e^{2x}\sin x - \int 2e^{2x}\sin x dx \\ &= e^{2x}\sin x + \int 2e^{2x} d\cos x \\ &= e^{2x}\sin x + 2e^{2x}\cos x - \int 2\cos x de^{2x} \\ &= e^{2x}\sin x + 2e^{2x}\cos x - \int 4e^{2x}\cos x dx\end{aligned}$$

So

$$\int e^{2x}\cos x dx = \frac{e^{2x}\sin x + 2e^{2x}\cos x}{5} + C$$

□

(9) (9 points) $\int \cos^4 x dx$

Solution.

$$\begin{aligned}\int \cos^4 x dx &= \int \left(\frac{\cos 2x + 1}{2}\right)^2 dx \\ &= \int \frac{\cos^2 2x + 2\cos 2x + 1}{4} dx \\ &= \int \frac{\frac{\cos 4x+1}{2} + 2\cos 2x + 1}{4} dx \\ &= \int \frac{\cos 4x}{8} + \frac{\cos 2x}{2} + \frac{3}{8} dx \\ &= \frac{\sin 4x}{32} + \frac{\sin 2x}{4} + \frac{3}{8}x + C\end{aligned}$$

□

(10) (9 points) $\int (\tan x + \sec x) \sin 2x dx$

Solution.

$$\begin{aligned}\int (\tan x + \sec x) \sin 2x dx &= \int \left(\frac{\sin x}{\cos x} + \frac{1}{\cos x} \right) 2 \sin x \cos x dx \\ &= \int 2\sin^2 x + 2 \sin x dx \\ &= \int (1 - \cos 2x) + 2 \sin x dx \\ &= x - \frac{\sin 2x}{2} - 2 \cos x + C\end{aligned}$$

□

(11) (9 points) $\int \frac{dx}{x^2 \sqrt{4 - x^2}}$

Solution. Set $x = 2 \sin \theta$,

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{4 - x^2}} &= \int \frac{2 \cos \theta}{4 \sin^2 \theta \sqrt{4 - 4 \sin^2 \theta}} d\theta \\ &= \int \frac{1}{4 \sin^2 \theta} d\theta \\ &= \int \frac{1}{4} \csc^2 \theta d\theta \\ &= -\frac{1}{4} \cot \theta + C \\ &= -\frac{1}{4} \cot \left(\arcsin \frac{x}{2} \right) + C\end{aligned}$$

□