MATH 122, FALL SEMESTER 2008

Exam 2

Name (print): ____________________________

INSTRUCTIONS

(1) Calculators are NOT allowed.
(2) Present your solutions in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

QN	PTS
1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

(1) (13 points) Evaluate the integral

$$\int \frac{2}{(x^2+2x+3)(x+1)}dx$$

Solution. Let

$$\frac{2}{(x^2+2x+3)(x+1)} = \frac{Ax+B}{(x^2+2x+3)} + \frac{C}{(x+1)} = \frac{(Ax+B)(x+1)+C(x^2+2x+3)}{(x^2+2x+3)(x+1)},$$

Therefore

$$A+C=0, A+B+2C=0, B+3C=2,$$

So

$$A=-1, B=-1, C=1,$$

$$\frac{2}{(x^2+2x+3)(x+1)} = -\frac{x+1}{(x^2+2x+3)} + \frac{1}{(x+1)}$$

$$\int \frac{2}{(x^2+2x+3)(x+1)}dx = \int -\frac{x+1}{(x^2+2x+3)} + \frac{1}{(x+1)}dx = \int -\frac{d(x^2+2x+3)}{2(x^2+2x+3)} + \int \frac{1}{x+1}dx$$

$$= -\frac{\ln(x^2+2x+3)}{2} + \ln|x+1| + C$$

□

(2) (13 points) Evaluate the integral

$$\int_0^2 \frac{1}{\sqrt{(x-1)}^3}dx$$

Solution.

$$\int_0^2 \frac{1}{\sqrt{(x-1)}^3}dx = \int_0^1 \frac{1}{\sqrt{(x-1)}^3}dx + \int_1^2 \frac{1}{\sqrt{(x-1)}^3}dx = \lim_{a\longrightarrow 1^-} \left.\frac{-2}{\sqrt{x-1}}\right|_0^a + \lim_{a\longrightarrow 1^+} \left.\frac{-2}{\sqrt{x-1}}\right|_a^2$$

So the improper integral does not exist. □

(3) (13 points) Evaluate the integral

$$\int_1^\infty \frac{\ln x}{x^2} dx$$

Solution.

$$\int_1^\infty \frac{\ln x}{x^2} dx = \int_1^\infty -\ln x d\frac{1}{x} = -\frac{\ln x}{x}\bigg|_1^\infty + \int_1^\infty \frac{1}{x^2} dx = \lim_{a\to\infty} -\frac{\ln x}{x}\bigg|_1^a - \lim_{a\to\infty} \frac{1}{x}\bigg|_1^a = 1$$

□

(4) (13 points) Evaluate the integral

$$\int_0^\infty \frac{e^{\frac{-1}{x}}}{x^2} dx$$

Solution.

$$\int_0^\infty \frac{e^{\frac{-1}{x}}}{x^2} dx = \int_0^\infty -e^{\frac{-1}{x}} d\frac{1}{x} = \lim_{a\longrightarrow\infty} e^{\frac{-1}{x}}\bigg|_1^a + \lim_{a\longrightarrow 0^+} e^{\frac{-1}{x}}\bigg|_a^1 = (1-1/e) + (1/e - 0) = 1$$

□

(5) (13 points) Find the length of the curve

$$y = \ln \sin x, \quad 0 \le x \le 1$$

Solution.

$$L = \int_0^1 \sqrt{1+(y')^2}dx = \int_0^1 \sqrt{1+\left(\frac{\cos x}{\sin x}\right)^2}dx = \int_0^1 \csc x dx = \lim_{a\to 0+} -\ln|\csc x + \cot x||_a^1 = \infty$$

So the curve is infinitely long. □

(6) (13 points) Use Euler's method with step size 0.5 to compute the approximate y-values y_1, y_2, and y_3 of the solution of teh initial value problem

$$xy' = 3 + xy, \quad y(1) = 1.$$

Solution. Since $y' = f(x, y) = \frac{3+xy}{x}$, the formula for Euler's method is

$$y_n = y_{n-1} + 0.5 f(x_{n-1}, y_{n-1}).$$

$$x_0 = 1, y_0 = 1$$
$$x_1 = 1.5, y_1 = 1 + 0.5\frac{3+1\cdot 1}{1} = 3$$
$$x_2 = 2, y_2 = 3 + 0.5\frac{3+1.5\cdot 3}{1.5} = 5.5$$
$$x_3 = 2.5, y_3 = 5.5 + 0.5\frac{3+2\cdot 5.5}{2} = 9$$

□

(7) (13 points) Find the solution to

$$\frac{dy}{dx} = \tan y + \cot y, \quad y(0) = \pi/4$$

Solution.

$$\frac{dy}{\tan y + \cot y} = dx$$
$$\cos y \sin y dy = dx$$

Taking integral on both sides, we get

$$\frac{1}{2}\sin^2 y = x + C, \text{ or } -\frac{1}{2}\cos^2 y = x + D$$

Plug in the intial value condition $y(0) = \pi/4$, we get $C = 1/4 = -D$. So the solution is

$$\frac{1}{2}\sin^2 y = x + 1/4, \text{ or } -\frac{1}{2}\cos^2 y = x - 1/4$$

□

(8) (13 points) Find the solution to

$$\frac{dy}{dx} = e^x + e^{x+y}$$

Solution.

$$\frac{dy}{dx} = e^x(1 + e^y)$$
$$\frac{dy}{1 + e^y} = e^x dx$$
$$\int \frac{dy}{1 + e^y} = \int e^x dx$$
$$\int \frac{de^y}{e^y(1 + e^y)} = \int e^x dx$$

Setting $z = e^y$ on the left-hand side, we have

$$\int \frac{dz}{z(1 + z)} = e^x + C$$
$$\int \left(\frac{1}{z} - \frac{1}{1 + z}\right) dz = e^x + C$$
$$\ln z - \ln(1 + z) = e^x + C$$
$$\ln e^y - \ln(1 + e^y) = e^x + C$$

or

$$\frac{e^y}{1 + e^y} = Ce^{e^x}$$

□