MATH 115, FALL SEMESTER 2008

Exam 3

Name (print): ____________________________

INSTRUCTIONS

(1) Calculators are allowed.
(2) Present your solutions in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

QN	PTS
1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

(1) (5 points) Find the Cartesian equation for the curve given in polar equation.

$$\sin\theta\cos\theta = 1$$

Solution.

$$r\sin\theta r\cos\theta = r^2$$
$$xy = x^2 + y^2$$

□

(2) (5 points) Find a polar equation for the curve.

$$x + y = 1$$

Solution.

$$r\sin\theta + r\cos\theta = 1$$
$$r(\sin\theta + \cos\theta) = 1$$

□

(3) (6 points) Given Cartesian coordiantes $(\sqrt{3}, 1)$ of a point, find polar coordinates (r, θ) of the point where

$$a)\ r > 0, 0 \leq \theta < 2\pi, \quad b)\ r < 0, 0 \leq \theta < 2\pi, \quad c)\ r > 0, -2\pi < \theta \leq 0$$

Solution.

$$a)\ (2, \pi/6), \quad b)\ (-2, 7\pi/6), \quad c)\ (2, -11\pi/6)$$

□

(4) A curve is given by

$$x = 1 - t^2, \quad y = \sin \pi t$$

(a) (10 points) Find the point at which the curve has a vertical tangent line?

Solution.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\pi \cos \pi t}{-2t}$$

So when $t = 0$, or at the point $(1, 0)$, the curve has vertical tangent line. □

(b) (10 points) Find the equations to the two tangent lines at $(0, 0)$

Solution. Since when $t = \pm 1$, the curve is at $(0, 0)$, it has two tangent lines at the point.
When $t = 1$, $\frac{dy}{dx} = \frac{-\pi}{-2} = \frac{\pi}{2}$, so the tangent line is $y = \frac{\pi}{2}x$.
When $t = -1$, $\frac{dy}{dx} = \frac{\pi}{-2} = -\frac{\pi}{2}$, so the tangent line is $y = -\frac{\pi}{2}x$. □

(5) Explain clearly whether the sequence converges or diverges. If it converges, find the limit.

(a) (10 points) $a_n = \frac{1}{n \ln n}$

Solution. Converges to 0 □

(b) (10 points) $a_n = \frac{(-1)^n (n^3 + 3n^2)}{2n^3 + 3n + 2}$

Solution. Diverges □

(6) (10 points) Find the length of the curve

$$x = 1 - t^2, \quad y = t^3, \quad 0 \le t \le 2$$

Solution. The curve length is

$$L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dx = \int_0^2 \sqrt{4t^2 + 9t^4} dx = \int_0^2 2t\sqrt{1 + \frac{9}{4}t^2} dx$$

$$= \int_0^2 \frac{4}{9}\sqrt{1 + \frac{9}{4}t^2} d\left(1 + \frac{9}{4}t^2\right) = \frac{4}{9}\frac{2}{3}\left(1 + \frac{9}{4}t^2\right)^{\frac{3}{2}}\Bigg|_2^0 = \frac{8}{27}(\sqrt{10} - 1)$$

□

(7) Evaluate

(a) (10 points) $\sum_{i=1}^{\infty} \frac{2+9^i}{10^i}$

Solution. $\sum_{i=1}^{\infty} \frac{2+9^i}{10^9} = \sum_{i=1}^{\infty} \frac{2}{10^9} + \sum_{i=1}^{\infty} \left(\frac{9}{10}\right)^9 = \frac{2}{10}\frac{10}{9} + \frac{9}{10}\frac{10}{1} = \frac{83}{9}$ □

(b) (10 points) $\sum_{i=1}^{\infty} \frac{3}{n(n+3)}$

Solution. $\frac{3}{n(n+3)} = \frac{1}{n} - \frac{1}{n+3}$ So

$$\sum_{i=1}^{\infty} \frac{3}{n(n+3)} = (1 - \frac{1}{4}) + (\frac{1}{2} - \frac{1}{5}) + (\frac{1}{3} - \frac{1}{6}) + \cdots = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

□

(8) Explain clearly whether the series is convergent or divergent. Do not evaluate.

(a) (10 points) $\sum_{n=1}^{\infty} ne^{-1/n}$

Solution. Since $\lim_{n\to\infty} ne^{-1/n} = \infty$, the series is divergent $\square$

(b) (10 points) $\sum_{n=1}^{\infty} ne^{-n}$

Solution. Let $f(x) = xe^{-x}$, and $f(n) = a_n$.

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x}$$

Hence $\int_1^{\infty} xe^{-x}dx = 2/e$ is convergent, and so does the series. $\square$