

Math 224 Exam 1

Name (print): _____

UIN: _____

Section: _____

- (1) Calculators can be used.
- (2) Present your solutions in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

QN	PTS
1	_____
2	_____
3	_____
4	_____
5	_____
6	_____
7	_____
8	_____
TOTAL	_____

Part I: Multiple choices

- (1) (10 points) Solve the following initial value problem

$$y' = \frac{x + xy}{2 + y}, \quad y(0) = 1.$$

Solution.

$$\begin{aligned} \frac{dy}{dx} &= x \frac{1 + y}{2 + y} \\ \frac{2 + y}{1 + y} dy &= x dx \\ \int 1 + \frac{1}{1 + y} dy &= \int x dx \\ y + \ln|1 + y| &= x^2/2 + C \end{aligned}$$

□

- (2) (10 points) Tank A holds 100 gal of pure water. Tank B holds 150 gal of solution in which is dissolved 20 lb of salt. At time $t = 0$, a solution containing 1 lb of salt per gallon begins to enter Tank A at the rate of 5 gallons per minute. At the same time a drain is opened at the bottom of the tank allowing the well-stirred solution to leave the tank at a rate of 3 gal/min, and enter Tank B at the same rate. A drain at the bottom of Tank B allows the solution to leave at a rate of 5 gal/min. Write down the initial value problem governing the amount of salt in both tanks at time t . DO NOT SOLVE.

Solution. Let t be time, $x(t), y(t)$ be the amount of salt in Tank A and B at time t , respectively.

Tank A:

$$x' = 5 - 3 \frac{x}{100 + 2t}, \quad x(0) = 0$$

Tank B:

$$y' = 3 \frac{x}{100 + 2t} - 3 \frac{y}{150 - 2t}, \quad y(0) = 20$$

□

- (3) (10 points) A bullet of mass $m = 0.1$ kg was shot up vertically from ground level with an initial vertical velocity of 1200 m/s. The air offers resistance proportional to the bullet's velocity ($k = -0.3$). Write down the initial value problem using time t and displacement x as independent variables respectively. DO NOT SOLVE.

Solution. Let upward be the positive direction, then by Newton's Law, the forces applied to the bullets are

$$mx'' = -mg + kx'$$

So the equation is

$$0.1x'' = -0.98 - 0.3x', \quad x(0) = 1200$$

□

- (4) (10 points) Compute 3 iterations of Euler's method with step size $h = 0.1$.

$$y' = y^2 - t, \quad y(0) = 1$$

Solution. Let $y' = f(t, y) = y^2 - t$, then Euler's Method says

$$t_{n+1} = t_n + h, \quad y_{n+1} = y_n + f(t_n, y_n)h$$

Therefore

$$\begin{aligned} t_0 &= 0, y_0 = 1 \\ t_1 &= 0, y_1 = y_0 + (y_0^2 - t_0)h = 1 + 0.1 = 1.1 \\ t_2 &= 0, y_2 = y_1 + (y_1^2 - t_1)h = 1.1 + ((1.1^2 - 0.1)0.1) = 1.22 \\ t_3 &= 0, y_3 = y_2 + (y_2^2 - t_2)h = 1.34884 \end{aligned}$$

□

- (5) Given the following initial value problem $\begin{cases} t \frac{dy}{dt} = -y + t^2 \\ y(1) = 1 \end{cases}$

- (a) (8 points) Find the solution to this problem.

Solution.

$$y' = f(t, y) = -\frac{y}{t} + t$$

Let $a(t) = -\frac{1}{t}$, $u(t) = e^{-\int a(t)dt} = t$, then

$$y(t) = \frac{1}{u(t)} \int u(t)t^2 dt = \frac{t^2}{3} + C.$$

Since $y(1) = 1$, $C = \frac{2}{3}$. So

$$y(t) = \frac{t^2 + 2}{3}.$$

□

- (b) (4 points) Find the interval of existence of the solution to this problem.

Solution. Since $f(t, y)$ is continuous for all real numbers except $t = 0$, and the initial point is at $(1, 1)$, the interval of existence is $(0, \infty)$. □

- (6) (10 points) Use the variation of parameters to find the general solution of

$$x' = x \tan(t) + \sin(t)$$

Hint: the solution to the homogeneous equation $x' = x \tan(t)$ is $x_h(t) = \frac{C}{\cos(t)}$. [No credit for any other method]

Solution. Let $x(t) = u(t)x_h(t) = \frac{u(t)}{\cos t}$. Plug the solution back into the differential equation, we get

$$x'(t) = \frac{u' \cos t + u \sin t}{\cos^2 t} = \frac{u(t)}{\cos t} \tan t + \sin t.$$

So

$$u' = \cos t \sin t, \\ u = \int \cos t \sin t dt = -\frac{\cos^2 t}{2} + C.$$

Therefore a general solution is

$$x(t) = \frac{u(t)}{\cos t} = -\frac{\cos t}{2} + \frac{C}{\cos t}$$

□

- (7) A population of fish (measured in thousands) in a pond, if left alone, is governed by the logistic equation

$$P' = 0.1P(1 - P/10),$$

Supposedly the pond had 7 (thousand) fish at the beginning.

- (a) (6 points) What is the long term projection of the fish population?

Solution. 10 thousand, since it is a stable equilibrium solution.

It is also obvious from the solution of the logistic solution

$$P = \frac{10C}{C + (10 - C)e^{-0.1t}},$$

which has a carrying capacity of 10.

□

- (b) (6 points) Recently recreational fishermen are catching 100 fish each day. Write down a differential equation that governs the rate of change of fish population. [DO NOT SOLVE]

Solution.

$$P' = 0.1P(1 - P/10) - 0.1$$

□

(8) Consider the autonomous equation

$$\frac{dy}{dt} = (y^2 - 4)(y + 3)$$

(a) (7 points) Find all the equilibrium equations.

Solution. Solving $(y^2 - 4)(y + 3) = 0$, we get the equilibrium equations

$$y = 2, \quad y = -2, \quad y = -3.$$

□

(b) (7 points) Sketch the phase line. Identify the asymptotically stable and unstable equilibrium points.

Solution. Stable at $y = -2$. Unstable at $y = -3, 2$.

□

(c) (7 points) Sketch the solution curves passing through $A(1, 2.5)$, $B(1, 1.5)$, $C(1, -2.5)$ and $D(1, -4)$ in ONE graph.

Solution.

□

(d) (5 points) Explain why the solution $y_1(t)$ that passes through the point A satisfies $y_1 > 2$ for all t .

Solution. Since $(y^2 - 4)(y + 3)$ is continuous, and its partial derivative is also continuous, all solutions to initial value problems are unique.

Now $y = 2$ is a solution to the differential equation. If $y_1(t_0) = 2$ for some t_0 , it has to be constantly 2, which is not true because $y_1(1) = 2.5$. So $y_1(t)$ can never be 2. Now that it started above 2, it has to be always on top of 2, because it is continuous.

□