Math 224 Exam 2

Name (print): ____________________________
Signature: ____________________________

(1) Calculators can be used.
(2) Present your solutions in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

QN	PTS
1	
2	
3	
4	
5	
6	
7	
8	
9	
TOTAL	

(1) (8 points) Find a particular solution to

$$y'' + y = t + e^t$$

Solution. Let

$$y = a + bt + ce^t,$$

and plug it into the system. We get

$$ce^t + a + bt + ce^t = t + e^t.$$

Therefore

$$a = 0, \quad b = 1, \quad 2c = 1,$$

so

$$y = t + e^t/2.$$

□

(2) (10 points) Find the solution to

$$y'' + 2y' + 2y = 5\cos t, \quad y(0) = 0, y'(0) = 0$$

Solution. The characteristic polynomial is

$$r^2 + 2r + 2 = 0,$$

with roots $r = \frac{-2\pm\sqrt{4-8}}{2} = -1 \pm i$. So a general solution to the system $y'' + 2y' + 2y = 0$ is $y = C_1e^{-t}\cos t + C_2e^{-t}\sin t$.

To find a particular solution, let $y = a\sin t + b\cos t$, and plug it into the system. We get

$$-a\sin t - b\cos t + 2a\cos t - 2b\sin t + 2a\sin t + 2b\cos t = 5\cos t.$$

Therefore $-2b+a = 0, \quad 2a+b = 5$, i.e., $a = 2, \quad b = 1$. So a particular solution is $y = 2\sin t+\cos t$. Hence a general solution to the problem is

$$y = 2\sin t + \cos t + C_1e^{-t}\cos t + C_2e^{-t}\sin t.$$

Since $y(0) = y'(0) = 0$, we have $1 + C_1 = 0, 2 - C_1 + C_2 = 0$, so $C_1 = -1, C_2 = -3$. It follows that the solution is

$$y = 2\sin t + \cos t - e^{-t}\cos t - 3e^{-t}\sin t$$

□

(3) (8 points)Find the Laplace transform for

$$f(t) = t^2 + e^{2t}\cos 3t - 1$$

Solution.

$$\mathfrak{L}\{f(t)\}(s) = \frac{2}{s^3} + \frac{s-2}{(s-2)^2+3^2} - \frac{1}{s} = \frac{-11s^2 - 8s + 26 + 2s^3}{s^3(s^2 - 4s + 13)}$$

□

(4) Find the inverse Laplace transfor for the following

(a) (8 points) $F(s) = \frac{s}{s^2+2s+5}$

Solution.

$$\frac{s}{s^2+2s+5} = \frac{s}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}$$

$$\mathfrak{L}^{-1}\{F(s)\}(t) = -e^t \cos 2t - \frac{e^{-t}\sin 2t}{2}$$

□

(b) (8 points) $F(s) = \frac{2s-2}{s^2-2s-3}$

Solution.

$$\frac{2s-2}{s^2-2s-3} = \frac{2s-2}{(s-3)(s+1)} = \frac{1}{s+1} + \frac{1}{s-3}$$

$$\mathfrak{L}^{-1}\{F\}(t) = e^{-t} + e^{3t}$$

□

(5) (8 points) Find the unit impulse response to the system

$$y'' - 3y + 2y = \delta(t), \quad y(0) = 0, y'(0) = 0$$

Solution. Applying Laplace transform on both sides, we get

$$\mathfrak{L}\{y\} = \frac{1}{s^2-3s+2} = \frac{1}{s-2} - \frac{1}{s-1}.$$

So the answer is

$$e(t) = e^{2t} - e^t.$$

□

(6) Let

$$f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < \infty \end{cases}$$

(a) (8 points) Redefine $f(t)$ using Heaviside function.

Solution. $f(t) = tH(t) + (1-t)H(t-1)$ □

(b) (8 points) Find the Laplace transform of $f(t)$.

Solution.

$$\mathfrak{L}\{f\} = \mathfrak{L}\{tH(t)\} - \mathfrak{L}\{(t-1)H(t-1)\} = \frac{1}{s^2} - e^{-s}\frac{1}{s^2}$$

□

(c) (8 points) Use Laplace transform to solve for $Y(s) = \mathfrak{L}\{y(t)\}$ (Do NOT solve for $y(t)$)

$$y'' - y = f(t), \quad y(0) = 1, \quad y'(0) = 1.$$

Solution. Applying Laplace transform on both sides, we get

$$s^2 Y(s) - y'(0) - sy(0) - Y(s) = \mathfrak{L}\{f(t)\},$$

so

$$Y(s) = \left((1 - e^{-s})\frac{1}{s^2} + (1+s) \right) \frac{1}{s^2 - 1}$$

□

(7) Given

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

find

(a) (4 points) $A + 2B$

Solution.

$$\begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix}$$

□

(b) (4 points) AB

Solution.

$$\begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix}$$

□

(c) (4 points) A^T

Solution.

$$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

□

(8) (8 points) Find the convolution of the functions

$$f(t) = t, \quad g(t) = e^{-t}$$

Solution.

$$f*g(t) = \int_0^t f(u)g(t-u)dt = \int_0^t ue^{u-t}dt = e^{-t}\int_0^t ue^u dt = e^{-t}\left(ue^u - e^u\right)\Big|_0^t = e^{-t}(te^t - e^t + 1) = t - 1 + e^{-t}$$

□

(9) (10 points) Find the solution to $Ax = b$, where

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & -1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Solution. The augmented matrix is

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & -2 & 2 \end{pmatrix}$$

Add the twice of the first row to the third, we get

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

Subtract the third row from the second, we get

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

Exchange the second and third row, we get a row echelon form

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

which translates into

$$\begin{aligned} -x_1 + x_2 + x_3 &= 0 \\ x_2 &= 2 \\ x_3 &= -1 \end{aligned}$$

so the solution is

$$x = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

□