MATH 320, Spring SEMESTER 2008

Exam 1

Name (print): ____________________________
Signature: ____________________________

INSTRUCTIONS

(1) Calculators are NOT allowed to use.
(2) Answers with factorials or powers are accepted, e.g., 10!, $9 \times 8 \times 7$ or $100 - 2^4$ are acceptable.
(3) Present your solutions in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

QN	PTS
1	
2	
3	
4	
5	
TOTAL	

(1) (10 points) Show that for any 5 integers, there are always 3 of them whose sum is divisible by 3.

Solution. There are only 3 possible remainders when integers are divided by 3: 0, 1, 2. If all of them show up when those 5 numbers are divided by 3, we add those 3. Otherwise, at most 2 of $\{0, 1, 2\}$ are available. By Pigeonhole Principle, 1 of them appear at least 3 times. Add those numbers up. □

(2) (10 points) Let $p = 23$, $q = 15$, find a and b such that $ap+bq = 1$. Use Chinese Remainder Thereom to find m such that $m \bmod p = 5$ and $m \bmod q = 9$.

Solution. $a = 2, b = -3, m = 9ap + 5bq = 189.$ □

(3) (10 points) The population of a small town is 20,000. If each resident has initials of 3 letters. Is it true there must be 2 residents witht the same initials?

Solution. The total population is $26^3 = 17576 < 20000$. So there must be 2 residents with the same initials. □

(4) (10 points) John started to do bussiness over the Internet. During a 4-week span, he made at least one sales each day. He also made no more than 10 sales each week. Show that there is a succession of days during which he made exactly 15 sales.

Solution. Let a_i be the number of total sales for the first i days. Then $1 \leq a_1 < \ldots < a_{28} \leq 40$. So $15 \leq a_1 + 15 < \ldots < a_{28} + 15 \leq 55$. There are a total of 56 such numbers, and at most 55 values, hence two of them are equal. Since a_i are all different, there must be a_m, a_n such that $a_n = a_m + 15$, i.e., the number of sales made from the $(m+1)^{th}$ day through n^{th} day is exactly 15. □

(5) Consider all the positive integers LESS THAN a million.

(a) (10 points) How many of them contain the digit 2?

Solution. Consider all 6-digit positive numbers that can have leading 0's. Total number of such numbers is $10^6 - 1 = 999999$, while the number of those without digit 2 is $9^6 - 1$. Therefore the answer is $10^6 - 9^6 = 468559$. □

(b) (10 points) How many of them contain either the digit 2 or 3?

Solution. The number of those without digits 2 and 3 is $8^6 - 1$. Therefore the answer is $10^6 - 8^6 = 737856$. □

(c) (10 points) How many of them start with 2 or end with 3?

Solution. The number of those start with 2 is 10^5. The number of those end with 3 is 10^5. The number of those start with 2 AND end with 3 is 10^4. So the answer is $10^5 + 10^5 - 10^4 = 190000$. □

(d) (10 points) How many of them contain either 2 or 3 but not 4?

Solution. Total number of such numbers without 4 is $9^6 - 1$, while the number of those without digit 2, 3 and 4 is $7^6 - 1$. Therefore the answer is $9^6 - 7^6 = 413792$. □

(e) (10 points) How many of them are even with different digits that start with either 1 or 2?

Solution. For the even numbers starting with 1, The number of choices for the first digit is 1, for the last is 5, so the answer is $1 \times 5 \times 8 \times 7 \times 6 \times 5 = 8400$.
For the even numbers starting with 2, the number of choices for the first digit is 1, for the last is 4, so the answer is $1 \times 4 \times 8 \times 7 \times 6 \times 5 = 6720$.
Therefore the total number is $8400 + 6720 = 15120$. □

(f) (10 points) How many of them are factors of $2^5 \times 3^3 \times 5^2 \times 7^2 = 1058400$?

Solution. The total number of factors of the number is $6 \times 4 \times 3 \times 3$. Out of these factors, only the original number itself is greater than 1 million. So the answer is $6 \times 4 \times 3 \times 3 - 1 = 215$. □