P40, 5: Show that if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 3n\}$, then there are always two which differ by at most 2.

First proof. Let the numbers be $a_1 < \ldots < a_{n+1}$. If any two of them differ by at least 3, then $a_{i+1} - a_i \geq 3$. Therefore $a_{n+1} - a_1 \geq 3n$, which is impossible because the greatest difference can be only $3n - 1$. □

Second proof. Divide the $3n$ numbers into n groups, $\{1, 2, 3\}, \{4, 5, 6\}, \ldots, \{3n-2, 3n-1, 3n\}$. Since we are picking $n+1$ numbers, at least two of them will be in the same group, which means the difference between the two is at most 2. □

P40, 15: Prove that, for any $n+1$ integers $a_1, a_2, \ldots, a_{n+1}$, there exist two of the integers a_i and a_j with $i \neq j$ such that $a_i - a_j$ is divisible by n.

Proof. Let b_i be the remainder of a_i when divided by n. So we have $n+1$ such numbers, but only n choices $\{0, 1, \ldots, n-1\}$ for the remainders. Hence two of numbers in $\{b_i\}$ have to be the same, say, b_i and b_j, then $a_i - a_j$ is divisible by n. □

Extra: What is the minimal number n such that if n integers are chosen, there always exist three of them whose sum is divisible by 3?

Proof. The answer is 5.

First, we show that if four numbers are chosen, it is possible that the sum of any three is not divisible by 3. For example, $\{1, 4, 5, 8\}$.

Now we need to prove that if five numbers $\{a_1, \ldots, a_5\}$ are chosen, there always exist three of them whose sum is divisible by 3.

Consider the remainders $\{b_1, \ldots, b_5\}$ of the five numbers when divided by 3. If all three possible remainders are in the list, say, $a_1 = 3k_1, a_2 = 3k_2 + 1, a_3 = 3k_2 + 2$, then $a_1 + a_2 + a_3$ is divisible by 3. Otherwise, we have at most two possible remainders available, but five numbers chosen, so at least three of the numbers will have the same remainder. The sum of these three numbers is divisible by 3. □