# MATH 320, Spring SEMESTER 2009

## Exam 1

Name (print): ____________________________

Signature: ____________________________

INSTRUCTIONS

(1) Calculators are NOT allowed.

(2) Answers with factorials or powers are accepted, e.g., 10!, $9 \times 8 \times 7$ or $100 - 2^4$ are acceptable.

(3) Present your solutions in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

| QN | PTS |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| TOTAL | |

(1) (17 points) Let $p = 38$, $q = 13$, find $a$ and $b$ such that $ap + bq = 1$. Use the Chinese Remainder Thereom to find $m$ such that $m \bmod p = 5$ and $m \bmod q = 9$.

*Solution.* $a = -1, b = 3, m = 9ap + 5bq = -147 = 347 \bmod 38 \cdot 17.$ □

(2) (17 points) Given 10 French books, 20 Spanish books, 8 German books, 15 Russian books, and 25 Italian books, how many books must be chosen to guarantee there are 12 books of the same language?

*Solution.* We need to pick all French and German books, so at least $10 + 11 + 8 + 11 + 11 + 1 = 52$ books. □

(3) (17 points) Show that any subset of $n$ distinct integers between 3 and $2n$ ($n \geq 2$) always contains a pair of integers that are relatively prime. (Hint: $a$ and $a+1$ do not have a common divisor other than 1)

*Solution.* Divide the number into pairs $(3,4),(5,6),\ldots,(2n-1,2n)$. There are $n-1$ pairs, and within each pair, the numbers are coprime. Since we are picking $n$ numbers, at least two of them are in the same pair. □

(4) (17 points) How many distinct factors does $2^5 \times 3^3 \times 5^2 \times 7^2 = 1058400$ have that are less than half a million?

*Solution.* The total number of factors of the number is $6 \times 4 \times 3 \times 3$. Out of these factors, only the original number itself and $\frac{1058400}{2}$ is greater than 1/2 million. So the answer is $6 \times 4 \times 3 \times 3 - 2 = 214$. □

(5) (17 points) In how many ways can we seat three men, three women, and two kids at a round table with 8 seats, so that the kids are not together. What if there are 9 seats?

*Solution.* If there are 8 seats, fix one of the children at the "top" of the table, then the other kid has 5 choices. There are 6! choices for the other men and woman. So the answer is $5 \cdot 6!$.

If there are 9 seats, let the empty seat be the "top". Then there are 8! ways to seat the people, while $2 \cdot 7!$ ways to seat the kids together, so the answer is $8! - 2 \cdot 7! = 6 \cdot 7!$. (You can also treat the empty seat as a person, and use the same idea as in 8 seats.) □

(6) (17 points) *Five* 52-card decks are shuffled together. How many orderings are there if all the cards of the same suit are together?

*Solution.* There are 4! ways to order the suits. For each suit, there are $\frac{65!}{5!^{13}}$ to order the cards. So the total number of ways is $4!\left(\frac{65!}{5!^{13}}\right)^4$. □